The crepant resolution conjecture for Donaldson–Thomas invariants via wall-crossing

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Voor mijn ouders,
en voor Dieuwke

در هر فلکی مردمکی می‌بینم

هر مردمکش را فلکی می‌بینم

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Lay Summary

The crepant resolution conjecture is a problem in *enumerative geometry* concerned with counting curves on a singular complex projective three-dimensional variety X called a *Calabi–Yau threefold*. This conjecture is motivated by both mathematics and theoretical physics. Let us first explain what it states.

Algebraic geometry is the field of mathematics concerned with solutions of polynomial equations. For example, one can study solutions to the equation

$$\text{E}\colon y^2 - x^2(x+1) = 0. \tag{0.0.1}$$

Examples of such solutions are the pairs $(x, y) = (0, 0)$ and $(x, y) = (-1, 0)$ as the reader can easily check by plugging in these numbers in the equation. This is the *algebraic* part of algebraic geometry. However, even though equation (0.0.1) is very simple – it has only two variables x and y, the powers of the variables are small, the coefficients are integers – it is impossible to write down all solutions to this equation.

This is where the *geometry* comes in. By plotting all points (x, y) on E, as in the figure below, we can study the properties of the shape they form.

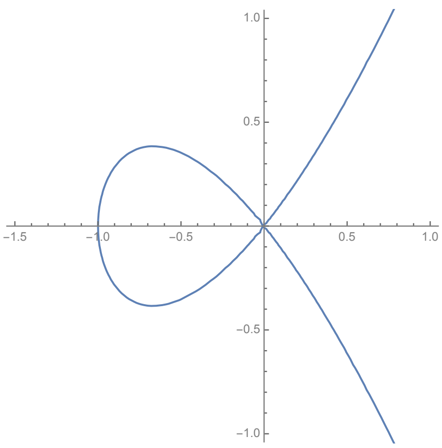

Figure 1: A plot of the points (x, y) on the curve E.

The geometry of E may teach us about the solutions to equation (0.0.1). For example, the plot shows that the solution $(0,0)$ is rather special, in fact, it is a *singular* point of the curve E. Similarly, there is a reflection symmetry in the x-axis by sending $y \mapsto -y$.

Enumerative geometry is a branch of algebraic geometry that is concerned with counting solutions to geometric problems. A simple example of a counting problem is:

How many straight lines pass through two given points in a plane?

Surely the answer is: 1. In other words, the *solution set contains one element*. However, in the *special* situations where the lines are parallel or the two given points are the same, the answers are different, namely 0 and ∞ respectively. *Generically*, there is a unique line and the answer is 1.

The crepant resolution conjecture concerns the problem of generically counting curves inside a very particular three-dimensional *complex* space X; as a *real* space it is six-dimensional, and is hard to picture. Projectivity is a certain compactness property that forces many counting problems to be finite, and prevents *special* situations like the parallel lines from occurring. The singularities of X satisfy a property known as Gorenstein, which means that they are relatively mild. Finally, a *curve* is a one-dimensional complex space, such as E in equation (0.0.1).

An accurate example of such X is given by two cones on top of each other joined at their apexes; or alternatively, a pinched cylinder of which the real part is shown in the figure below. We call the pinched point P. As one intuitively sees, the point P is different from the others. Indeed, P is the only point where X is *singular*. Note that X is only singular in a 'small' set, which is part of the reason why we call the singularities of X relatively mild. Although this picture is two-dimensional, it extends to a proper three-dimensional example by taking $\mathrm{X} \times \mathbf{P}^1$, where $\mathbf{P}^1$ is the complex projective line.

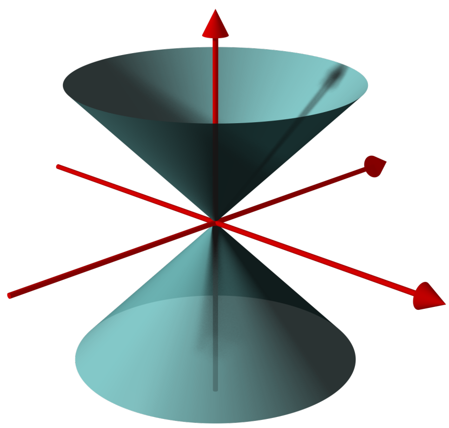

Figure 2: Two cones joined at their apexes, or a pinched cylinder.

Now that we have a picture in mind, how do we count curves in this context? First of all, we should specify what *type* of curve we wish to count. In our elementary example, we asked for *straight* lines through two points, not circles or more eccentric shapes. For each type of curve, we want to know how many of them live in X.

Unfortunately, this count is not so easy to define. A technical construction called a *virtual fundamental class* is required. Very roughly, this construction tells you how many elements the 'right' solution set has. Counterintuitively, this answer can be negative!

But there is a more important problem: this construction does not work well for singular varieties. There are two different strategies to circumvent this problem.

1. Classical: we replace the singular space X by a *non-singular* approximation Y of X and count curves there. This is called a *resolution of singularities* of X. One can think of such a resolution as a smoothing of the singularities. It is important that Y be as close to X as possible; this is where the *crepancy* condition comes in.

In the example of the double cone, a resolution would be pulling apart the cone point to (re)obtain a cylinder. There is a resolution morphism relating the singular space to its resolution. In our picture, it is the projection map that pinches one of the cylinder's circles to the apex. Thus outside the circle and P respectively, the spaces 'are the same'.

2. Stacky: we work with the singular space X but we remember how it is obtained by pinching one of the smooth cylinder's circles. Mathematically, this new object $\mathcal{X}$ is not a space but a *stack*. More importantly, as far as stacks go, $\mathcal{X}$ is non-singular!

It is hard to picture how this stack looks like, since it is not really a space. Roughly speaking, we imagine it looks like X with 'extra structure' at the point P. There is a map $\pi\colon \mathcal{X} \to \mathrm{X}$ forgetting this extra structure. The following diagram summarises the setup.

$$\begin{tikzcd} \mathcal{X} \arrow[dr, "\pi"'] & & \mathrm{Y} \arrow[dl, "f"] \\ & \mathrm{X} & \end{tikzcd} \tag{0.0.2}$$

Since both $\mathcal{X}$ and Y are non-singular, the technical construction of a virtual fundamental class can be performed on both spaces. We thus obtain counts of curves on $\mathcal{X}$ and Y, and both should represent the correct count of curves on X.

However, the above example reveals an important issue: we have removed a *point* from X, but added in a *curve* C to obtain Y! In order to get a 'correct' count of curves on X, we should count the curves on the resolution Y but subtract the curves we added to resolve the singularity. Note that curves intersecting C in finitely many points are no problem, since those are simply curves running through the singular point P.

Similarly, the extra structure at P on the stack $\mathcal{X}$ alters the count.

The crepant resolution conjecture states how these count are related. The original formulation of J. Bryan, C. Cadman, and B. Young in [BCY12] conjectured that there is a dictionary between types of curves on $\mathcal{X}$ and types of curves on Y such that their counts (in $\mathcal{X}$ and Y respectively) are equal.

In Chapter 3 of this thesis, I provide a counterexample to this dictionary. The further work of Chapters 4 and 5, the latter joined with John Calabrese and Jørgen Rennemo, establishes a *reinterpretation* of this conjecture which we now explain.

Suppose that for every integer $n \in \mathbf{Z}$ we have a type of curve, such as straight lines ($n = 1$), circles ($n = 2$), and so on. We write the counting invariants as

$$a_n(\mathrm{Y}) := \text{ number of curves of type } n \text{ on } \mathrm{Y}, \tag{0.0.3}$$

and similarly $b_n(\mathcal{X})$ for the number of curves of type n on $\mathcal{X}$; recall that $a_n(\mathrm{Y})$ and $b_n(\mathcal{X})$ are integers themselves. Next comes the great trick: we package this information in a *generating series.* This means that we write

$$\mathrm{A}(x) = \sum_{n\in\mathbf{Z}} a_n(\mathrm{Y})x^n \quad \text{and} \quad \mathrm{B}(x) = \sum_{m\in\mathbf{Z}} b_m(\mathcal{X})x^m, \tag{0.0.4}$$

where x is a variable that keeps track of the type of curve its coefficient counts. The original claim of the conjecture is that $a_n(\mathrm{Y})$ and $b_n(\mathcal{X})$ are equal. This is not true.

We prove that *only the entire functions* $\mathrm{A}(x)$ *and* $\mathrm{B}(x)$ *can be related to each other.* Let us consider a simple example of such behaviour. Consider the generating functions

$$\begin{aligned} \mathrm{A}(x) &= 1 + x + x^2 + x^3 + \ldots = \sum_{n\geq 0} x^n \\ \mathrm{B}(x) &= -x^{-1} - x^{-2} - x^{-3} + \ldots = -\sum_{m\geq 1} x^{-m} \end{aligned} \tag{0.0.5}$$

Clearly, $a_n(\mathrm{Y}) \neq b_n(\mathcal{X})$ because the former vanish when $n < 0$ whereas the latter vanish when $n \geq 0$. However, both $\mathrm{A}(x)$ and $\mathrm{B}(x)$ are (expansions of) a *rational function* called the *geometric series.* Indeed, we have

$$\mathrm{A}(x) = \frac{1}{1-x} = -x^{-1}\left(\frac{1}{1-x^{-1}}\right) = \mathrm{B}(x). \tag{0.0.6}$$

As such, one can extract all curve counts on Y provided one knows *all of them* on $\mathcal{X}$, and vice versa. In this thesis, we prove that the crepant resolution conjecture (only) holds in this sense, as an equality of rational functions.

Abstract

Let Y be a smooth complex projective Calabi–Yau threefold. Donaldson–Thomas invariants [Tho00] are integer invariants that virtually enumerate curves on Y. They are organised in a generating series DT(Y) that is interesting from a variety of perspectives. For example, well-known series in mathematics and physics appear in explicit computations. Furthermore, closer to the topic of this thesis, the generating series of birational Calabi–Yau threefolds determine one another [Cal16a].

The crepant resolution conjecture for Donaldson–Thomas invariants [BCY12] conjectures another such comparison result. It relates the Donaldson–Thomas generating series of a certain type of three-dimensional Calabi–Yau orbifold to that of a particular resolution of singularities of its coarse moduli space. The conjectured relation is an equality of generating series.

In this thesis, I first provide a counterexample showing that this conjecture cannot hold as an equality of generating series. I then verify that both generating series are the Laurent expansion about different points of the same rational function. This suggests a reinterpretation of the crepant resolution conjecture as an equality of rational functions.

Second, following a strategy of Bridgeland [Bri11] and Toda [Tod10a, Tod13, Tod16a], I prove a wall-crossing formula in a motivic Hall algebra relating the Hilbert scheme of curves on the orbifold to that on the resolution. I introduce the notion of pair object associated to a torsion pair, putting ideal sheaves and stable pairs on the same footing, and generalise the wall-crossing formula to this setting, essentially breaking the former in many pieces. Pairs, and their wall-crossing formula, are fundamentally objects of the bounded derived category of the Calabi–Yau orbifold.

Finally, I present joint work with J. Calabrese and J. Rennemo [BCR] in which we use the wall-crossing formula and Joyce's integration map to prove the crepant resolution conjecture for Donaldson–Thomas invariants as an equality of rational functions. A crucial ingredient is a result of J. Rennemo that detects when two generating functions related by a wall-crossing are expansions of the same rational function.

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text. Where the work was done in collaboration with others, I have made a significant contribution. In particular, Chapter 5 is joint work with John Calabrese and Jørgen Rennemo.

This work has not been submitted for any other award or professional qualification.

Sjoerd Viktor Beentjes
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List of Publications

- S. Beentjes, J. Calabrese, and J. Rennemo, *A proof of the Crepant Resolution Conjecture for Donaldson–Thomas invariants*, in preparation.

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Chapter 1

The enumerative geometry of threefolds

In the past 25 years or so, the enumerative geometry of curves has become one of the richest topics in modern algebraic geometry, propelling the development of a host of new techniques with widespread applications in geometry proper. An excellent account of, and introduction to, many aspects of curve-counting theory can be found in [PT14]. We will, however, present some of these features, relevant to this work, in the following sections. This is followed by a discussion of the main results in this thesis, as well as future directions.

1.1 Introduction

Let us briefly discuss what is meant by *counting curves*. Enumerating curves in an algebraic variety Y proceeds by choosing a suitable compactification of the moduli space of curves in the variety, to then take a certain invariant of it. To have this invariant be well-defined, one typically specifies the numerical class of the curves being enumerated so as to obtain a moduli space of finite type.

Ideally, this compactification $\mathcal{M}$ is a zero-dimensional space. If not, geometric conditions (so-called *insertions*) can be imposed to cut down to such a setting. Then, the degree $\deg[\mathcal{M}]$ of the fundamental class $[\mathcal{M}]$ in the Chow ring $\mathrm{A}_*(\mathcal{M})$ is taken as the *count* of the number of curves on Y. In practice, however, such moduli spaces are highly singular and of positive dimension so this procedure is ill-defined. The existence of a *virtual* fundamental class $[\mathcal{M}]^{\mathrm{vir}} \in \mathrm{A}_0(\mathcal{M})$, representing the idealised zero-dimensional geometry, allows one to *virtually* enumerate curves on Y. Morally, this is viewed as the correct way of counting curves, and the curve-count is defined[1] as

$$\int_{[\mathcal{M}]^{\mathrm{vir}}} 1 = \deg[\mathcal{M}]^{\mathrm{vir}} \in \mathbf{Q}. \tag{1.1.1}$$

The rapid development of the enumerative geometry of curves has largely been due to the construction of virtual fundamental classes on a number of compactifications of moduli spaces of curves, following [LT98] and [BF97]. Moreover, there has been a fruitful exchange between the enumerative geometry of curves in threefolds on the one hand, and string theory, currently a popular contender for a unifying theory of the standard model and Einstein's general relativity, on the other hand.

In fact, the theory of curve-counting is richest if Y is a threefold. Furthermore, no insertions are needed when Y has the *Calabi–Yau* property.

Definition 1.1.1. A Calabi–Yau variety is a complex algebraic variety Y with trivialised canonical bundle $\omega_{\mathrm{Y}} \cong \mathcal{O}_{\mathrm{Y}}$ such that $\mathrm{H}^1(\mathrm{Y}, \mathcal{O}_{\mathrm{Y}}) = 0$.

There are two distinct flavours of Calabi–Yau threefolds: global and local ones.

Example 1.1.2. Arguably the most famous compact Calabi–Yau threefold is the Fermat quintic, defined by $\mathrm{Y} = \{x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0\} \subset \mathbf{P}^4$. Its canonical bundle is trivial by adjunction and $\mathrm{H}^1(\mathrm{Y}, \mathcal{O}_{\mathrm{Y}}) = 0$ follows by a cohomology computation.

Example 1.1.3. Let C be a smooth projective curve, and let E be a rank two vector bundle on C such that $\det(\mathrm{E}) = \omega_{\mathrm{C}}$. Then $\mathrm{Y} = \mathrm{Tot}(\mathrm{E}) \to \mathrm{C}$ satisfies $\omega_{\mathrm{Y}} \cong \mathcal{O}_{\mathrm{Y}}$. Similarly, if S is a smooth projective surface, then $\mathrm{Y} = \mathrm{Tot}(\omega_{\mathrm{S}}) \to \mathrm{S}$ has trivial canonical bundle.

For Y to be a Calabi–Yau threefold, we also need $\mathrm{H}^1(\mathrm{Y}, \mathcal{O}_{\mathrm{Y}}) = 0$. By Kodaira vanishing, this holds for $\mathrm{C} = \mathbf{P}^1$ and for S a del Pezzo or K3 surface respectively.

We will discuss and compare counts of curves on various Calabi–Yau varieties. For the remainder of this chapter, let Y be a smooth projective Calabi–Yau threefold.

1.1.1 What is a curve?

Essentially, there are two ways to describe a non-singular embedded curve $\mathrm{C} \subset \mathrm{Y}$:

(i) as an algebraic morphism $f\colon \mathrm{C} \to \mathrm{Y}$, so $\mathcal{O}_{\mathrm{Y}} \to f_* \mathcal{O}_{\mathrm{C}}$;

(ii) as the zero locus of an ideal of algebraic functions on Y, so $\mathrm{I}_{\mathrm{C}} \subset \mathcal{O}_{\mathrm{Y}}$.

Equivalently, a curve can be seen as a parametrised object with a map or as an unparametrised object with an embedding.

[1] Although the degree of a proper scheme is an integer, some curve-counting theories give rise to stacky compactifications of the moduli space of curves, in the sense of Deligne–Mumford. Their degree is only rational, in general. An example is the theory of Gromov–Witten invariants.

Example 1.1.4. It is not possible to restrict ones attention to non-singular curves, as their moduli are not compact in general. Consider the following local example. Let $\mathrm{C}_t = \{xy - t = 0\} \subset \mathbf{A}^2_{x,y} \times \mathbf{A}^1_t \to \mathbf{A}^1_t$ be a flat family of curves in $\mathbf{A}^2$ parametrised by the affine line. If $t \neq 0$ the curve C_t is non-singular, yielding a one-parameter family of non-singular curves in the moduli space $\mathcal{M}$ of curves on $\mathbf{A}^2$. But the limit of this family as $t \to 0$ is reducible, it has a nodal singularity at $(0,0) \in \mathbf{A}^2$, so $\mathcal{M}$ is not compact.

Note that compactifying the ambient space $\mathbf{A}^2 \subset \mathbf{P}^2$ does not resolve this issue.

We consider three compactifications of the moduli space of non-singular curves. To construct any moduli space as an algebraic object we need a notion of *family* of objects, and to have it be of finite type we need to fix the *numerical invariants* of the objects. For curves, it suffices to fix a homology class $\beta = [\mathrm{C}] \in \mathrm{H}_2(\mathrm{Y}, \mathbf{Z})$ and either[2] the arithmetic genus $g = p_a(\mathrm{C})$ or the holomorphic Euler characteristic $n = \chi(\mathcal{O}_\mathrm{C})$. We consider three notions of curve, and the corresponding notion of family of curves. Throughout, S denotes a complex scheme parametrising said family.

1. The first notion is that of a *stable map*. Curves are viewed as parametrised by an algebraic map $f\colon \mathrm{C} \to \mathrm{Y}$ such that $f_*[\mathrm{C}] = \beta$ and the arithmetic genus of C is g. In order to compactify this space, curves are allowed to have nodal singularities and the maps are allowed to degenerate in a certain way. More precisely

$$\overline{\mathcal{M}}_g(\mathrm{Y}, \beta) = \left\{ f\colon \mathrm{C} \to \mathrm{Y} \;\middle|\; \begin{array}{c} \text{C a connected projective nodal} \\ \text{curve of arithmetic genus } g, \\ f_*[\mathrm{C}]{=}\beta, \text{ and } \mathrm{Aut}(f) \text{ finite} \end{array} \right\} \tag{1.1.2}$$

   is called the moduli space of stable maps. The automorphism group of f is the subgroup $\mathrm{Aut}(f) \subset \mathrm{Aut}(\mathrm{C})$ of automorphisms ϕ of C such that $f = f \circ \phi$. It is proven in [Kon95] that $\overline{\mathcal{M}}_g(\mathrm{Y}, \beta)$ is a proper Deligne–Mumford stack, assuming that Y is a projective variety. By the valuative criterion for properness, this precisely means that limits as in the above example exist and are unique.

   An S-family of stable maps of class (β, g) is a flat proper morphism $\pi\colon \mathcal{C} \to \mathrm{S}$ together with a morphism $f\colon \mathcal{C} \to \mathrm{Y}$ such that each geometric fibre $f_s\colon \mathcal{C}_s \to \mathrm{X}$ is a stable map of curve class $f_{s,*}[\mathcal{C}_s] = \beta$ and arithmetic genus $p_a(\mathcal{C}_s) = g$.

2. The second notion is that of an *embedded curve*. Here, a curve is viewed as an unparametrised object with an embedding into Y, i.e., an embedded curve is a closed subscheme $\mathrm{C} \subset \mathrm{Y}$ of dimension one. The *Hilbert scheme* compactifies embedded curves by allowing degenerations to arbitrary at most one-dimensional closed subschemes. In some sense, this is a 'large' compactification, the necessity

[2] Recall that these determine each other, since $p_a(\mathrm{C}) = 1 - \chi(\mathcal{O}_\mathrm{C})$, i.e., $g = 1 - n$.

of which is shown in the classic example of a family of twisted cubics below. More precisely

$$\mathrm{Hilb}_n(\mathrm{Y},\beta) = \{\mathcal{O}_\mathrm{Y} \twoheadrightarrow \mathcal{O}_\mathrm{C} \,|\, [\mathrm{C}] = \beta,\ \chi(\mathcal{O}_\mathrm{C}) = n\}, \tag{1.1.3}$$

and an S-family of curves is an S-flat quotient $\mathcal{O}_{\mathrm{S}\times\mathrm{Y}} \twoheadrightarrow \mathcal{O}_\mathcal{Z}$ such that for each geometric fibre $[\mathrm{Z}_s] = \beta$ and $\chi(\mathcal{O}_{\mathrm{Z}_s}) = n$; if the variety Y is not projective we require the schematic support of the quotient $\mathcal{O}_\mathcal{Z}$ to be proper over S.

In the fundamental [Gro95], Grothendieck has proven that the functor

$$\underline{\mathrm{Hilb}}_n(\mathrm{Y},\beta)\colon \mathrm{Sch}/\mathbf{C} \to \mathrm{Set}, \quad \mathrm{S} \mapsto \left\{ \mathcal{O}_{\mathrm{S}\times\mathrm{Y}} \twoheadrightarrow \mathcal{O}_\mathcal{Z} \,\middle|\, \begin{array}{c} \mathcal{O}_\mathcal{Z} \text{ is flat over S,} \\ [\mathcal{Z}_s]=\beta,\, \chi(\mathcal{O}_{\mathcal{Z}_s})=n \end{array} \right\}.$$

is represented by a projective variety we denote by $\mathrm{Hilb}_n(\mathrm{Y},\beta)$.

3. The third, more modern, notion of curve we consider is that of a *stable pair*, introduced by Pandharipande–Thomas in [PT09]. A stable pair (F, s) consists of a one-dimensional sheaf F of class $[\operatorname{supp}\mathrm{F}] = \beta$ and $\chi(\mathrm{Y},\mathrm{F}) = n$, together with a section $s \in \mathrm{H}^0(\mathrm{Y},\mathrm{F})$. This data satisfies two stability requirements:

   (i) the sheaf F is pure, i.e., any non-trivial subsheaf is one-dimensional,

   (ii) the cokernel of the section s is zero-dimensional.

   Two stable pairs (F, s) and (F', s') are said to be isomorphic if there exists an isomorphism $\phi\colon \mathrm{F} \to \mathrm{F}'$ commuting with the sections: $\phi \circ s = s'$.

   The purity of F implies that the schematic support C_F of F is a Cohen–Macaulay curve. Indeed, the section s factors through its image, inducing an exact sequence

$$0 \to \mathcal{O}_{\mathrm{C}_\mathrm{F}} \to \mathrm{F} \to \mathrm{Q}_\mathrm{F} \to 0. \tag{1.1.4}$$

   Here $\mathrm{Q}_\mathrm{F} = \operatorname{coker}(s)$ is zero-dimensional, and supported on the curve C_F.

   Examples of stable pairs are obtained from an embedded Cohen–Macaulay curve $\mathrm{C} \subset \mathrm{Y}$ and a Cartier divisor $\mathrm{D} \subset \mathrm{C}$. The associated stable pair is

$$(\mathcal{O}_\mathrm{C}(\mathrm{D}), s_\mathrm{D}\colon \mathcal{O}_\mathrm{Y} \twoheadrightarrow \mathcal{O}_\mathrm{C} \hookrightarrow \mathcal{O}_\mathrm{C}(\mathrm{D})), \tag{1.1.5}$$

   and the cokernel of s_D is the zero-dimensional sheaf $\mathcal{O}_\mathrm{D}(\mathrm{D})$. In fact, if the underlying curve C_F is *Gorenstein* (for example, non-singular) then every stable pair schematically supported on C_F is of the form (1.1.5); see [PT10, Prop. B.5].

   An S-family of stable pairs is a pair $(\mathrm{F}, s\colon \mathcal{O}_{\mathrm{S}\times\mathrm{Y}} \to \mathrm{F})$ on $\mathrm{S}\times\mathrm{Y}$ such that F is flat over S and the restriction (F_t, s_t) to the fibre $\{t\}\times\mathrm{Y}$ is a stable pair for every closed point $t \in \mathrm{S}$. With this notion of family, a projective moduli space of stable pairs is constructed by Le Potier in [LP93] via GIT methods. We denote it by

$$\mathrm{P}_n(\mathrm{Y},\beta) = \left\{ \mathcal{O}_\mathrm{Y} \xrightarrow{s} \mathrm{F} \,\middle|\, \substack{\mathrm{F}\text{ is pure}, \dim(\mathrm{F})=1, \dim(\operatorname{coker} s)=0 \\ \chi(\mathrm{F})=n, [\operatorname{supp}\mathrm{F}]=\beta} \right\} \tag{1.1.6}$$

The following remark ties in well with our notion of *pair* in section 4.1.2. By [PT09], a stable pair (F, s) determines and is determined by its associated complex

$$\mathrm{I} = \{\mathcal{O}_\mathrm{Y} \xrightarrow{s} \mathrm{F}\} \in \mathrm{D}^b(\mathrm{Y}) \tag{1.1.7}$$

with trivial determinant in the derived category of Y. Moreover, these notions are equivalent up to all orders in deformation theory. This shows that the moduli space of stable pairs is a component of the moduli space of complexes in $\mathrm{D}^b(\mathrm{Y})$.

Remark 1.1.5. The Hilbert scheme and the moduli space of stable pairs are isomorphic along the open locus of embedded Cohen–Macaulay curves, but differ in their boundary.

We illustrate the difference between embedded curves and stable pairs in the following example of a flat family of rational twisted cubics in $\mathbf{P}^3$; see [PS85] for further details.

Example 1.1.6. A rational twisted cubic is a smooth projective non-planar curve $\mathrm{C} \subset \mathbf{P}^3$ of genus zero and degree three. All such are projectively equivalent to the image of

$$f\colon \mathbf{P}^1 \to \mathbf{P}^3, \quad (x:y) \mapsto (x^3 : x^2y : xy^2 : y^3). \tag{1.1.8}$$

By [PS85, Lem. 1,2], there exists a flat family of such cubics parametrised by $0 \neq t \in \mathbf{A}^1$ whose flat limit as $t \to 0$ is a nodal planar cubic that is non-reduced at the node.

As discussed in [PT14, p. 19], it is easier to consider the following local model. Let $\mathrm{C}_t = \{x = 0 = z\} \sqcup \{y = 0 = z - t\} \subset \mathbf{A}^3_{x,y,z} \times \mathbf{A}^1_t \to \mathbf{A}^1_t$ denote the flat family (for $t \neq 0$) parametrising the y-axis in the $z = 0$ plane and the x-axis in the $z = t$ plane. By Remark 1.1.5, this family defines a $\mathbf{C}^\times$-valued point of the Hilbert scheme *and* of the stable pair moduli space, as it lands in their open common locus. Both these schemes are projective, and we now illustrate how they differ at their boundary by comparing the respective limits of C_t as $t \to 0$.

1. The flat limit: the ideal of the family of curves C_t in $\mathbf{A}^3 \times \mathbf{A}^1_t$ is given by

$$\mathrm{I}_t = (x,z)\cdot(y,z-t) = (xy, x(z-t), yz, z(z-t)) \leq \mathbf{C}[x,y,z,t]. \tag{1.1.9}$$

Taking the flat limit as $t \to 0$ yields the ideal $\mathrm{I}_0 = (xy, xz, yz, z^2) \subset (xy, z)$. Thus the limit in the Hilbert scheme, the corresponding embedded curve, is the intersection of the x and y-axes in the plane $z = 0$ with a non-reduced node 'pointing upwards' out of the node; this point can break off in a further flat family. In short, the flat limit is given by the node $\{xy = 0 = z\} \subset \mathbf{A}^3$ with an embedded point at the origin and, as such, it is non-reduced.

2. The stable pair limit: we distinguish the two components of the family C_t by writing $\mathrm{C}_t^1 = \{x = 0 = z\}$ for the y-axis and $\mathrm{C}_t^2 = \{y = 0 = z - t\}$ for the family of x-axes. Write $\mathrm{Y} = \mathbf{A}^3_{x,y,z}$. For $t \neq 0$ we have surjections

$$s_t\colon \mathcal{O}_{\mathrm{Y}} \twoheadrightarrow \mathcal{O}_{\mathrm{C}_t^1} \oplus \mathcal{O}_{\mathrm{C}_t^2}. \tag{1.1.10}$$

Simply putting $t = 0$ in the defining ideal yields the stable pair limit

$$s_0\colon \mathcal{O}_{\mathrm{Y}} \to \mathcal{O}_{\mathrm{C}_0^1} \oplus \mathcal{O}_{\mathrm{C}_0^2}, \tag{1.1.11}$$

where $\mathrm{C}_0^1 \sqcup \mathrm{C}_0^2$ is the intersection of the x-axis and the y-axis in the plane $z = 0$. Note that s_0 is not surjective at the origin since $\operatorname{coker}(s_0) = \mathcal{O}_0$, but this is no problem since we are working with stable pairs. In short, the limit stable pair is the normalisation of the image $\{xy = 0 = z\} \subset \mathbf{A}^3$ of the node (in the plane $z = 0$).

1.1.2 Gromov–Witten vs. Donaldson–Thomas

With the compactifications of the moduli space of non-singular curves in place, we may start counting curves. The existence of a suitable *perfect obstruction theory* in the sense of Behrend–Fantechi [BF97] implies the existence of a virtual fundamental class. Roughly speaking, a perfect obstruction theory is a two-term complex of vector bundles equipped with a comparison morphism to the truncated cotangent complex that encodes deformations of curves and their obstructions. In each of the three cases, such an obstruction theory has been constructed.

If a proper moduli space $\mathcal{M}$ has a perfect obstruction theory, its associated fundamental class $[\mathcal{M}]^{\mathrm{vir}}$ is an element of $\mathrm{A}_{\mathrm{vdim}}(\mathcal{M})$ where vdim is the *virtual* or *expected dimension*. If vdim = 0, the degree of $[\mathcal{M}]^{\mathrm{vir}}$ is the associated virtual count. This happens, for example, when the obstruction theory is *symmetric*, i.e., the deformations and obstructions are dual to each other.

1. The moduli space $\overline{\mathcal{M}}_g(\mathrm{Y}, \beta)$ of stable maps admits a perfect obstruction theory by [BF97], which has virtual dimension

$$d^{\mathrm{GW}}_{\beta,g} = \int_\beta c_1(\mathrm{Y}) + (\dim_{\mathbf{C}} \mathrm{Y} - 3)(1 - g). \tag{1.1.12}$$

In particular, if Y is a Calabi–Yau threefold we have $d^{\mathrm{GW}}_{\beta,g} = 0$ for all (β, g). The associated invariants are called (connected) *Gromov–Witten invariants*, and denoted

$$\mathrm{GW}_{\mathrm{Y}}(\beta, g) = \int_{[\overline{\mathcal{M}}_g(\mathrm{Y},\beta)]^{\mathrm{vir}}} 1 \in \mathbf{Q}. \tag{1.1.13}$$

Note that by the stacky nature of the moduli space of stable maps the invariants are only rational in general. We collect these invariants in a generating series

$$\mathrm{Z}^{\mathrm{GW}}_{\beta}(\mathrm{Y}; u) = \sum_{g\geq 0} \mathrm{GW}_g(\mathrm{Y}, \beta) u^{2g-2}. \tag{1.1.14}$$

2. The Hilbert scheme of curves on a Calabi–Yau threefold is isomorphic to the moduli space of torsion free rank one sheaves of trivial determinant. More precisely, there is an isomorphism of schemes

$$\Phi\colon \mathrm{Hilb}_n(\mathrm{Y}, \beta) \longrightarrow \mathrm{I}_n(\mathrm{Y}, \beta), \quad [\mathcal{O}_{\mathrm{Y}} \twoheadrightarrow \mathcal{O}_{\mathrm{C}}] \mapsto \mathrm{I}_{\mathrm{C}}, \tag{1.1.15}$$

where an S-valued point of $\mathrm{I}_n(\mathrm{Y}, \beta)$ is an S-flat sheaf $\mathcal{I}$ on $\mathrm{S} \times \mathrm{Y}$ such that $\mathcal{I}_s$ is a torsion free rank one sheaf with trivial determinant of Chern character $(1, 0, -\beta, -n)$ for each geometric point $s \in \mathrm{S}$. The latter space has a natural symmetric obstruction theory by virtue of the Calabi–Yau threefold property combined with Serre duality, namely

$$\mathrm{Ext}^1(\mathrm{I}_{\mathrm{C}}, \mathrm{I}_{\mathrm{C}})_0 \cong \mathrm{Ext}^2(\mathrm{I}_{\mathrm{C}}, \mathrm{I}_{\mathrm{C}})^*_0, \tag{1.1.16}$$

where $(-)_0$ denotes the trace-free part. Indeed, let $\mathcal{I}_{\mathcal{C}}$ on $\mathrm{Y} \times \mathrm{I}_n(\mathrm{Y}, \beta)$ be the universal ideal sheaf of class $(1, 0, -\beta, -n)$, and let $\pi\colon \mathrm{Y} \times \mathrm{I}_n(\mathrm{Y}, \beta) \to \mathrm{I}_n(\mathrm{Y}, \beta)$ denote the projection. The Atiyah class of $\mathcal{I}_{\mathcal{C}}$ equips the two-term[3] complex

$$\mathbf{R}\pi_* \, \mathbf{R}\underline{\mathrm{Hom}}(\mathcal{I}_{\mathcal{C}}, \mathcal{I}_{\mathcal{C}})_0[2] \tag{1.1.17}$$

with the structure of a symmetric perfect obstruction theory on the moduli space of embedded curves by [Tho00]; this structure is explained in [HT10]. The associated invariants are called *Donaldson–Thomas invariants*, and are denoted by

$$\mathrm{DT}_n(\mathrm{Y}, \beta) = \int_{[\mathrm{I}_n(\mathrm{Y},\beta)]^{\mathrm{vir}}} 1 \in \mathbf{Z}. \tag{1.1.18}$$

[3] We claim that any such sheaf I_{C} satisfies $\mathrm{Hom}(\mathrm{I}_{\mathrm{C}}, \mathrm{I}_{\mathrm{C}})_0 = 0$, and hence $\mathrm{Ext}^3(\mathrm{I}_{\mathrm{C}}, \mathrm{I}_{\mathrm{C}})_0 = 0$ by Serre duality. This follows because I_{C} is torsion free of rank one. Indeed, let $f\colon \mathrm{I}_{\mathrm{C}} \to \mathrm{I}_{\mathrm{C}}$ be a non-zero morphism. Since I_{C} is torsion free, $\mathrm{rk}\,\mathrm{im}(f) = 1$, whence it follows that f is injective. But $\mathrm{ch}(\ker f) = \mathrm{ch}(\mathrm{coker}\, f)$ implies that f is surjective too. It follows that $\mathrm{Hom}(\mathrm{I}_{\mathrm{C}}, \mathrm{I}_{\mathrm{C}}) = \mathbf{C} \cdot \mathbf{1}$.

Again, we collect these invariants in a generating series

$$Z_\beta^{\mathrm{DT}}(Y;q) = \sum_{n\in\mathbf{Z}} \mathrm{DT}_n(Y,\beta)q^n. \tag{1.1.19}$$

Remark 1.1.7. The Donaldson–Thomas generating series is a Laurent series in q. Indeed, for a fixed curves class β, the Hilbert scheme is empty for small enough Euler characteristic n. This can be deduced for example from the "add on a floating point" trick, as can be found in [Tod09, Lem. 3.10], as follows.

Suppose $\mathrm{Hilb}_{n-k}(Y,\beta)$ is not empty for $k > 0$, and let $C \subset Y$ be an embedded curve of $\chi(\mathcal{O}_C) = n-k$ and $[C] = \beta$. By adding on k floating points to C we infer

$$\dim \mathrm{Hilb}_n(Y,\beta) \geq 3k \tag{1.1.20}$$

contradicting the boundedness of the Hilbert scheme. The claim follows.

3. In [HT10], a perfect obstruction theory is constructed on certain moduli spaces of complexes, extending the approach of [Tho00]. The moduli space of stable pairs $P_n(Y,\beta)$ is a fine moduli scheme [LP93], so there exists a universal stable pair

$$\mathcal{I} = \{\mathcal{O}_{P_n(Y,\beta)\times Y} \xrightarrow{s} \mathcal{F}) \tag{1.1.21}$$

in the derived category of $P_n(Y,\beta)\times Y$. In the CY3 case, in analogy to (1.1.17), a natural perfect obstruction theory on $P_n(Y,\beta)$ is given by the two-term complex

$$\mathbf{R}\pi_* \mathbf{R}\underline{\mathrm{Hom}}(\mathcal{I},\mathcal{I})_0[2] \tag{1.1.22}$$

induced by the Atiyah class of $\mathcal{I}$. Here $\pi\colon Y \times P_n(Y,\beta) \to P_n(Y,\beta)$ denotes the natural projection. Again by Serre duality, this theory is symmetric. The associated invariants, called *stable pair invariants*, are denoted by

$$\mathrm{PT}_n(Y,\beta) = \int_{[P_n(Y,\beta)]^{\mathrm{vir}}} 1 \in \mathbf{Z}. \tag{1.1.23}$$

As before, we collect these invariants in a generating series

$$Z_\beta^{\mathrm{PT}}(Y;q) = \sum_{n\in\mathbf{Z}} \mathrm{PT}_n(Y,\beta)q^n. \tag{1.1.24}$$

This series is a Laurent series in q by similar arguments as for $Z_\beta^{\mathrm{DT}}(Y;q)$. Indeed, any stable pair $(s\colon \mathcal{O}_Y \to F)$, of class $[F] = \beta$ and $\chi(F) = n$, factors through a surjection $s\colon \mathcal{O}_Y \twoheadrightarrow \mathcal{O}_{C_F} \coloneqq \mathrm{im}(s)$ leading to the exact sequence in equation (1.1.4).

The claim now follows from $n = \chi(\mathrm{F}) \geq \chi(\mathcal{O}_{\mathrm{C_F}})$, $[\mathrm{C_F}] = \beta$, and Remark 1.1.7.

These counting invariants are, in fact, *deformation invariant* in the following sense. Roughly speaking, given a flat family $p\colon \mathcal{X} \to \mathrm{C}$ of Calabi–Yau threefolds $\mathrm{X}_t = p^{-1}(t)$ parametrised by a smooth curve C, one can construct a cycle on $\mathcal{X}$ that restricts to the correct degree zero virtual fundamental classes on the fibres. For Donaldson–Thomas theory this is [Tho00, Cor. 3.53], and for Pandharipande–Thomas theory this is [PT09, Thm. 2.15]. By the principle of *conservation of numbers* of [Ful98, § 10.2], the degree of these cycles, i.e., the invariants, is constant.

Finally, the fundamental work of K. Behrend [Beh09] shows that DT and PT invariants may be computed as topological Euler characteristics weighted by a certain constructible function now known as the *Behrend function.* Although this approach is not manifestly deformation invariant, it allows for the use of cut-and-paste techniques since the topological Euler characteristic e is a *motivic* function, i.e.,

$$e(\mathrm{X}) = e(\mathrm{Z}) + e(\mathrm{X}\backslash\mathrm{Z}) \tag{1.1.25}$$

where X is a topological space and $\mathrm{Z} \subset \mathrm{X}$ is a closed subset. These techniques are essential in constructing the integration map on the motivic Hall algebra as described in section 2.3.3, which is in turn fundamental in proving comparison theorems such as the DT/PT correspondence; see 1.2.9. Furthermore, Behrend's work allows for the introduction of DT and PT invariants for non-proper varieties, such as the local Calabi–Yau threefolds of example 1.1.3.

1.2 Questions in curve counting

Given the various theories to count curves on Calabi–Yau threefolds introduced in the previous section, there are a number of questions that naturally arise. Roughly speaking, we may categorize these questions as follows. Let Y be a smooth projective Calabi–Yau threefold, and let $\beta \in \mathrm{H}_2(\mathrm{Y}, \mathbf{Z})$ be a curve class.

1. *Computation*: given a specific choice of Y and a specific choice of curve-counting theory, can we determine certain invariants, or even complete generating functions?
2. *Comparison I*: are the generating functions $\mathrm{Z}_\beta^{\mathrm{GW}}(\mathrm{Y};u)$, $\mathrm{Z}_\beta^{\mathrm{DT}}(\mathrm{Y};q)$, and $\mathrm{Z}_\beta^{\mathrm{PT}}(\mathrm{Y};q)$ equal, or do they determine each other in some way?
3. *Comparison II*: suppose Y' is another smooth projective Calabi–Yau threefold that is 'related' to Y, e.g., they are birational $\mathrm{Y} \sim_{\mathrm{bir}} \mathrm{Y}'$. Are the generating functions of Y and Y' equal, or do they determine each other in some way?
4. *Symmetry*: are the generating functions arising from counting curves Laurent expansions of rational functions? Do these functions have symmetries or modularity properties and, if so, are they dependent on the chosen Calabi–Yau threefold Y?
5. *Generalisation*: is it possible to extend the definitions of these curve-counting invariants? For instance, is it possible to drop the smoothness assumption for Donaldson–Thomas and Pandharipande–Thomas invariants?

We briefly summarise the status of these questions, where we place an emphasis on the sheaf-counting theories of Donaldson–Thomas and Pandharipande–Thomas, as the results of the current thesis fall in this realm. In the next section, we state the crepant resolution conjecture for Donaldson–Thomas invariants, as formulated by [BCY12], and indicate why it involves questions of types 2. through 5. in the above list.

1.2.1 Computations

On a general Calabi–Yau threefold Y, the generating series $\mathrm{Z}_\beta^{\mathrm{DT}}(q)$ of Donaldson–Thomas invariants is notoriously difficult to compute. The main difficulty lies in dealing with contributions from floating points. Indeed, given an embedded curve $\mathrm{C} \subset \mathrm{Y}$ of class $[\mathrm{C}] = \beta$ and $\chi(\mathcal{O}_\mathrm{C}) = n$ contributing to $\mathrm{Z}_\beta^{\mathrm{DT}}(\mathrm{Y};q)$, we obtain many more contributing curves

$$\mathcal{O}_\mathrm{Y} \twoheadrightarrow \mathcal{O}_{\mathrm{C}_{p_1,\dots,p_r}} = \mathcal{O}_\mathrm{C} \oplus \mathcal{O}_{p_1} \oplus \dots \oplus \mathcal{O}_{p_r} \tag{1.2.1}$$

of class $(\beta, n + \sum_{i=1}^r l(p_i))$ by simply adding $r \geq 1$ disjoint floating points $p_i \notin \mathrm{C}$ of length $l(p_i) \geq 1$. Thus, the DT generating series for each curve class $\beta \in \mathrm{H}_2(\mathrm{Y}, \mathbf{Z})$ probes the global geometry of Y in a complicated way; however, for local contributions see [BB07].

Zero-dimensional DT invariants

Embedded curves of class $\beta = 0$ are simply configurations of n points in Y. Thus

$$\mathrm{Hilb}_n(\mathrm{Y}, 0) = \mathrm{Y}^{[n]}$$

is the Hilbert scheme of n points on Y. As part of the famous MNOP conjectures (see section 1.2.2) Maulik, Nekrasov, Okounkov, and Pandharipande conjectured an explicit description of the generating function $\mathrm{Z}_0^{\mathrm{DT}}(\mathrm{Y}; q)$ of point-counts on Y.

Conjecture 1.2.1. [MNOP06, Conj. 1] Let Y be a smooth projective Calabi–Yau threefold. Then

$$\mathrm{Z}_0^{\mathrm{DT}}(\mathrm{Y}; q) = \mathrm{M}(-q)^{e(\mathrm{Y})}, \tag{1.2.2}$$

where $e(\mathrm{Y})$ is the topological Euler characteristic of Y and $\mathrm{M}(q) = \prod_{k \geq 1}(1 - q^k)^{-k}$ is the *MacMahon function*, the generating series for three-dimensional plane partitions.

In words, the degree zero partition function only depends on the topology of the Calabi–Yau threefold. By a result of Batyrev [Bat99], birational Calabi–Yau's have equal Euler characteristic. In particular, $\mathrm{Z}_0^{\mathrm{DT}}(\mathrm{Y}; q)$ is a birational invariant.

This conjecture has been proven in three different ways in [Li06, BF08, LP09]. We mention that the approach of Behrend–Fantechi in [BF08], via the Behrend function, allows a generalisation of the statement to quasi-projective non-singular threefolds.

Local DT and PT invariants

One motivation for introducing Pandharipande–Thomas invariants, where point contributions are localised on the support of the sheaf, is dealing with the issue of floating points. As a result, stable pair theory is much more amenable to computations.

We record the following explicit, and surprisingly instructive, example for later use.

Example 1.2.2. Let $\mathrm{Y} = \mathrm{Tot}(\mathcal{O}(-1) \oplus \mathcal{O}(-1)) \to \mathbf{P}^1$ be the quasi-projective CY3 often referred to as *local* $\mathbf{P}^1$ or the *resolved conifold*, and let $\mathbf{P}^1 \cong \mathrm{C} \hookrightarrow \mathrm{Y}$ denote its zero section. We want to compute the full stable pair generating function

$$\mathrm{Z}_{[\mathrm{C}]}^{\mathrm{PT}}(\mathrm{Y}; q) = \sum_{n \in \mathbf{Z}} \mathrm{PT}_n(\mathrm{Y}, [\mathrm{C}]) q^n \tag{1.2.3}$$

for the class $[\mathrm{C}]$ of the zero section.

Let $s \colon \mathcal{O}_{\mathrm{Y}} \to \mathrm{F}$ be a stable pair with $[\mathrm{F}] = [\mathrm{C}]$ and $\chi(\mathcal{O}_{\mathrm{Y}}, \mathrm{F}) = n$. Because F is pure of dimension one and $[\mathrm{C}]$ is an irreducible[4] curve class, the exact sequence

$$0 \to \mathrm{I_C} \cdot \mathrm{F} \to \mathrm{F} \to \mathrm{F}/\mathrm{I_C} \cdot \mathrm{F} \to 0 \tag{1.2.4}$$

implies that $\mathrm{I_C} \cdot \mathrm{F} = 0$, i.e., that F is schematically supported on C. Since C is non-singular, it follows that F is a locally free sheaf on C. But since $\dim \operatorname{coker}(s) \leq 0$ it must have rank one, whence $\mathrm{F} = \mathcal{O}_\mathrm{C}(k)$ for some $k \in \mathbf{Z}$. In fact, we have $k = n-1$ by $\chi(\mathcal{O}_\mathrm{X}, \mathrm{F}) = n$.

Thus, any stable pair of class $([\mathrm{C}], n)$ on Y is of the form $s \colon \mathcal{O}_\mathrm{Y} \to \mathcal{O}_\mathrm{C}(n-1)$. It immediately follows that $\mathrm{P}_n(\mathrm{Y}, [\mathrm{C}]) = \varnothing$ for $n \leq 0$, whence $\mathrm{PT}_n(\mathrm{Y}, [\mathrm{C}]) = 0$. For $n \geq 1$, the zero section of $\mathcal{O}_\mathrm{C}(n-1)$ is disqualified as its cokernel is one-dimensional. Moreover, scaling the section defines equivalent stable pairs. In conclusion

$$\mathrm{P}_n(\mathrm{Y}, [\mathrm{C}]) = \left(\mathrm{H}^0(\mathrm{X}, \mathcal{O}_\mathrm{C}(n-1)) - \{0\}\right)/\mathbf{C}^* \cong \mathbf{P}^{n-1} \tag{1.2.5}$$

for $n \geq 1$. Equivalently, since all stable pairs supported on a non-singular curve C are of the form $(\mathcal{O}_\mathrm{C}(\mathrm{D}), s_\mathrm{D})$ for an effective Cartier divisor $\mathrm{D} \subset \mathrm{C}$, we have

$$\mathrm{P}_n(\mathrm{Y}, [\mathrm{C}]) \cong \mathrm{Sym}^{n-1}(\mathbf{P}^1) \cong \mathbf{P}^{n-1}. \tag{1.2.6}$$

Working on a smooth scheme, we may use Behrend's Theorem 2.2.8. This yields

$$\mathrm{PT}_n(\mathrm{Y}, [\mathrm{C}]) = e_\mathrm{B}(\mathbf{P}^{n-1}) = (-1)^{\dim \mathbf{P}^{n-1}} e(\mathbf{P}^{n-1}) = (-1)^{n-1} n. \tag{1.2.7}$$

In conclusion, we find that $\mathrm{Z}^{\mathrm{PT}}_{[\mathrm{C}]}(\mathrm{Y}; q)$ is the expansion of a rational function

$$\mathrm{Z}^{\mathrm{PT}}_{[\mathrm{C}]}(\mathrm{Y}; q) = \sum_{n \geq 1} (-1)^{n-1} n q^n \sim_0 \frac{q}{(1+q)^2} =: f^{\mathrm{PT}}_{[\mathrm{C}]}(\mathrm{Y}; q), \tag{1.2.8}$$

where the symbol $\sim_0$ indicates expansion about $q = 0$; of course, in the ring of formal power series $\mathbf{Z}[[q]]$ the power series and the rational function are equal.

Finally, note that this rational function has the symmetry $f^{\mathrm{PT}}_{[\mathrm{C}]}(\mathrm{Y}; q) = f^{\mathrm{PT}}_{[\mathrm{C}]}(\mathrm{Y}; q^{-1})$.

Remark 1.2.3. The description in equation (1.2.6) directly generalises to the contribution to the PT generating series of any smooth infinitesimally isolated curve $\mathrm{C} \subset \mathrm{Y}$ of genus g, where Y is a smooth projective CY3; see [PT09, § 4.2]. One finds

$$\mathrm{Z}^{\mathrm{PT}}_{[\mathrm{C}]}(\mathrm{Y}; q) = q^{1-g}(1+q)^{2g-2}, \tag{1.2.9}$$

and this rational function is again invariant under $q \leftrightarrow q^{-1}$.

[4]An effective curve class $\beta \geq 0$ is irreducible if it cannot be written as $\beta = \beta_1 + \beta_2$ for $\beta_i \geq 0$.

Remark 1.2.4. Problems with floating points notwithstanding, for some geometries the DT partition function has been determined explicitly using degeneration techniques and virtual localisation with respect to torus actions. See for example [OP10].

1.2.2 Comparison theorems

We discuss various comparison conjectures and theorems between different types of invariants, or on different Calabi–Yau threefolds.

The GW/DT correspondence

Arguably the most famous of the comparison theorems, the GW/DT correspondence was originally conjectured by MNOP in [MNOP06, Conj. 3]. It essentially states that the Gromov–Witten and Donaldson–Thomas generating series determine each other after a variable change and a non-trivial analytic continuation.

More precisely, let

$$\mathrm{F}'(\mathrm{Y}; u, v) = \sum_{\beta \neq 0} \mathrm{Z}_{\beta}^{\mathrm{GW}}(\mathrm{Y}; u) v^{\beta} = \sum_{\beta \neq 0} \sum_{g \geq 0} \mathrm{GW}_g(\mathrm{Y}, \beta) u^{2g-2} v^{\beta}$$

denote the *reduced* Gromov–Witten potential of Y, omitting the constant maps for which $\beta = 0$. The exponential reduced partition function

$$\tilde{\mathrm{Z}}^{\mathrm{GW}}(\mathrm{Y}; u, v) = \exp \mathrm{F}'(\mathrm{Y}; u, v) \tag{1.2.10}$$

generates *disconnected* Gromov–Witten invariants of Y excluding constant map contributions. In turn, its expansion

$$\tilde{\mathrm{Z}}^{\mathrm{GW}}(\mathrm{Y}; u, v) = 1 + \sum_{\beta \neq 0} \tilde{\mathrm{Z}}_{\beta}^{\mathrm{GW}}(\mathrm{Y}; u) v^{\beta} \tag{1.2.11}$$

defines the *reduced* generating function $\tilde{\mathrm{Z}}_{\beta}^{\mathrm{GW}}(\mathrm{Y}; u)$ counting curves of class $\beta \neq 0$.

Similarly, as we are interested in *curve*-counts, we form the reduced DT generating series by formally quotienting out point contributions. To do so, set

$$\mathrm{Z}^{\mathrm{DT}}(\mathrm{Y}; q, z) = \sum_{\beta} \mathrm{Z}_{\beta}^{\mathrm{DT}}(\mathrm{Y}; q) z^{\beta} = \sum_{\beta} \sum_{n \in \mathbf{Z}} \mathrm{DT}_n(\mathrm{Y}, \beta) q^n z^{\beta}, \tag{1.2.12}$$

where we sum over all classes $\beta \in \mathrm{H}_2(\mathrm{Y}, \mathbf{Z})$. The reduced generating series is

$$\tilde{\mathrm{Z}}^{\mathrm{DT}}(\mathrm{Y}; q, z) = \mathrm{Z}^{\mathrm{DT}}(\mathrm{Y}; q, z)/\mathrm{Z}_0^{\mathrm{DT}}(\mathrm{Y}; q) = 1 + \sum_{\beta \neq 0} \tilde{\mathrm{Z}}_{\beta}^{\mathrm{DT}}(\mathrm{Y}; q) z^{\beta}. \tag{1.2.13}$$

We are now in a position to state the GW/DT correspondence.

Conjecture 1.2.5. [MNOP06, Conj. 2,3] The following statements hold for each $\beta \neq 0$.

1. The reduced series $\tilde{Z}^{\mathrm{DT}}_{\beta}(Y;q)$ is the expansion of a rational function in q invariant under the transformation $q \leftrightarrow q^{-1}$.
2. The change of variables $e^{iu} = -q$ equates the reduced generating series

$$\tilde{Z}^{\mathrm{GW}}_{\beta}(Y;u) = \tilde{Z}^{\mathrm{DT}}_{\beta}(Y;-e^{iu}). \tag{1.2.14}$$

Quoting [PT14, p. 23], the GW/DT correspondence should be viewed as involving an analytic continuation and series expansion about two different points: $q = 0$ or $u \to i\infty$ for DT invariants, and $q = -1$ or $u = 0$ for GW invariants.

Remark 1.2.6. Note that the change of variables in part (2) is well-defined by the result of part (1). Indeed, one can analytically continue a rational function and expand it about $q = -1$. Moreover, the Gromov–Witten invariants being *real* implies that its series should be invariant under the substitution $e^{iu} \mapsto e^{-iu}$. But this is precisely the invariance $q \leftrightarrow q^{-1}$ of $\tilde{Z}^{\mathrm{DT}}_{\beta}(Y;q)$ under the GW/DT correspondence.

Example 1.2.7. Let us examine the effect of the correspondence on the generating function $Z^{\mathrm{PT}}_{[C]}(Y;q)$ of local $\mathbf{P}^1$ found in example 1.2.2. Setting $q = -e^{iu}$ yields

$$f^{\mathrm{PT}}_{[C]}(Y;-e^{iu}) = \frac{-e^{iu}}{(1-e^{iu})^2} = \frac{1}{(e^{iu/2}-e^{-iu/2})^2} = (2\sin(u/2))^{-2}. \tag{1.2.15}$$

Expanding this series about $u = 0$, corresponding to $q = -1$, yields the GW invariants

$$Z^{\mathrm{GW}}_{[C]}(Y;u) = u^{-2} + \frac{1}{12} + \frac{1}{240}u^2 + \mathcal{O}(u^4). \tag{1.2.16}$$

Above are displayed the genus zero, one, and two contributions respectively.

Remark 1.2.8. Note that all but the genus zero term can be directly extracted from the expansion of the *series* $Z(q) \coloneqq Z^{\mathrm{PT}}_{[C]}(Y;q)$ about $q = 0$, corresponding to $u \to i\infty$. Indeed, writing $Z(q) = \sum_n a_n q^n$ then expanding $Z(-e^{iu})$ about $u = 0$ yields

$$\begin{aligned} Z(-e^{iu}) &= \sum_{n\in\mathbf{Z}} (-1)^n a_n (e^{iun}) = \sum_n (-1)^n a_n \left(1 + (in)u - \frac{n^2}{2}u^2 - \frac{in^3}{6}u^3 + \dots\right) \\ &= \left(\sum_n (-1)^n a_n\right) + i\left(\sum_n (-1)^n n a_n\right)u - \left(\sum_n (-1)^n \frac{n^2}{2!} a_n\right)u^2 - i\left(\sum_n (-1)^n \frac{n^3}{3!} a_n\right)u^3 + \dots \\ &= \sum_{g\geq 0} c_g u^g \end{aligned}$$

where $c_0 = \sum_n (-1)^n a_n = \mathrm{Z}(-1)$, where $c_1 = i\sum_n (-1)^n n a_n = -i\mathrm{Z}'(-1)$, and so on for $n \geq 0$. Denote the corresponding rational function by $f(q) = q(1+q)^{-2}$. Note that the symmetry $f(-e^{iu}) = f(-e^{-iu})$ implies that all odd coefficients vanish, i.e., $c_{2k+1} = 0$ for all $k \geq 0$; in particular, the resulting function is real-valued. It follows that we may write the generating function as

$$\mathrm{Z}(-e^{iu}) = \sum_{g \geq 1} b_g u^{2g-2}, \tag{1.2.17}$$

which is almost the usual form of Gromov–Witten invariants; the $b_0 u^{-2}$ term is missing.

It is not possible to evaluate the rational function $f(q)$, or its derivatives, at $q = -1$. However, in this case, analytic continuation of the zeta function allows us to attribute a finite value to the series $\{b_g \mid g \geq 1\}$. This is done as follows.

1. In equation (1.2.7) we found that $a_n = (-1)^{n-1} n$ for all $n \geq 0$ and zero otherwise.
2. It follows that the genus one coefficient yields
$$b_1 = \sum_{n \geq 0} (-1)^n (-1)^{n-1} n = -\sum_{n \geq 0} n = -\zeta(-1) = \frac{1}{12}.$$
3. We explicitly check that the coefficients c_{2k-1} vanish. For the first one we find
$$c_1 = i\sum_{n \geq 0} (-1)^n n (-1)^{n-1} n = -i\sum_{n \geq 0} n^2 = -i\zeta(-2) = 0$$
as required. In fact, $c_{2k-1} \sim \zeta(-2k) = 0$ for $k \geq 1$, so all odd coefficients vanish.
4. For the genus two coefficient, we find
$$c_2 = -\sum_{n \geq 0} (-1)^n \frac{n^2}{2} (-1)^{n-1} n = \frac{1}{2}\sum_{n \geq 0} n^3 = \frac{1}{2}\zeta(-3) = \frac{1}{240}.$$

Indeed, this reconstructs the first few terms of the Gromov–Witten partition function of local $\mathbf{P}^1$ as in equation (1.2.16). It is not strange that the genus zero term $b_0 u^{-2} = u^{-2}$ is missing, as the Laurent series expansion about $u = i\infty$ (that is, $q = 0$) cannot tell us anything about the pole at $u = 0$ (that is, $q = -1$).

This information *is* captured, however, by resumming the series expansion $\mathrm{Z}(q)$ of the rational function $f(q)$ about $q = 0$, then analytically continuing $f(q)$, and finally re-expanding the resulting function about $q = -1$.

The change of variables, part two of Conjecture 1.2.1, has been proven for non-singular, quasi-projective, toric threefolds in [MOOP11], and more recently for Calabi–Yau complete intersections in products of projective spaces in [PP17]; this includes the famous Fermat quintic of Example 1.1.2. Note that both papers are a real *tour de force*, building on many previous results and computations. In particular, they prove far more general statements about counting invariants relative to a non-singular divisor $\mathrm{S} \subset \mathrm{X}$ that reduce to the above one in the Calabi–Yau situation, i.e., the setting of this thesis.

The rationality statement and functional equation, part one of Conjecture 1.2.1, follow from the correspondence between Donaldson–Thomas and Pandharipande–Thomas invariants for Calabi–Yau threefolds, as discussed in the next two sections.

The DT/PT correspondence

As mentioned, floating point contributions probing the global geometry of Y make the DT partition functions notoriously difficult to compute. On a formal level, passing to the reduced partition functions $\tilde{\mathrm{Z}}_\beta^{\mathrm{DT}}(\mathrm{Y}; q)$ as in equation (1.2.13) deals with this problem.

A main motivation for the introduction of stable pair invariants in [PT09] is the conjecture that their counts geometrically realise the reduced DT partition function. More precisely, the generating function of counts of stable pairs of curve class β is precisely the reduced DT partition function. This is the DT/PT correspondence.

Conjecture 1.2.9. [PT09, Conj. 3.2] Let Y be a smooth projective Calabi–Yau threefold, and let $\beta \in \mathrm{H}_2(\mathrm{Y}, \mathbf{Z})$ be a curve class on Y. There is an equality of Laurent series

$$\mathrm{Z}_\beta^{\mathrm{PT}}(\mathrm{Y}; q) = \tilde{\mathrm{Z}}_\beta^{\mathrm{DT}}(\mathrm{Y}; q). \tag{1.2.18}$$

Remark 1.2.10. As an equality of Laurent series, we may multiply both sides of this equation by $\mathrm{Z}_0^{\mathrm{DT}}(\mathrm{Y}; q)$ and collect terms. For a fixed power of q, the statement reads

$$\sum_{m \in \mathbf{Z}} \mathrm{PT}_\beta(\mathrm{Y}, m) \cdot \mathrm{DT}_0(\mathrm{Y}, n - m) = \mathrm{DT}_\beta(\mathrm{Y}, n). \tag{1.2.19}$$

Note that the summation is in fact finite: it is bounded below since $\mathrm{PT}_\beta(\mathrm{Y}, m) = 0$ for $m \ll 0$, and it is bounded above because $\mathrm{DT}_0(\mathrm{Y}, n - m) = 0$ for $m > n$.

This result was for the first time rigorously interpreted as a wall-crossing formula in terms of (polynomial) stability conditions by A. Bayer [Bay09]. A reformulation in terms of weak stability conditions allowed Y. Toda to prove this conjecture in [Tod10a] modulo a certain structure result of the moduli stack of objects in the heart of any bounded t-structure on the bounded derived category of a Calabi–Yau threefold. Finally, in [Bri11] T. Bridgeland gave a full proof of the DT/PT correspondence using D. Joyce's motivic Hall algebra. The desired structure result has now been proven in [Tod16a] via work of [AHR15], also completing Y. Toda's proof.

A local version of the DT/PT correspondence, singling out contributions of a fixed curve, were proven by A. Ricolfi for smooth curves in [Ric17b, Ric17a], and later for all Cohen–Macaulay curves by G. Oberdieck in [Obe16]. The former exploits the motivic nature of DT invariants, using stratifications of the moduli spaces, whereas the latter proceeds by refining T. Bridgeland's arguments.

1.2.3 Rationality and symmetry

Through the DT/PT correspondence, the rationality and symmetry of the reduced DT series translates into rationality of the PT generating series and symmetry under $q \leftrightarrow q^{-1}$ of its rational re-summation. The precise statement is the following.

Conjecture 1.2.11. Let Y be a smooth projective Calabi–Yau threefold, and let $\beta \in \mathrm{H}_2(\mathrm{Y}, \mathbf{Z})$ be a curve class on Y. The series $\mathrm{Z}^{\mathrm{PT}}_\beta(\mathrm{Y}; q)$ is the Laurent expansion about $q = 0$ of a rational function $f_\beta(q)$ invariant under $q \leftrightarrow q^{-1}$, i.e., $f_\beta(q) = f_\beta(q^{-1})$.

Note that only the rational function $f_\beta(q)$ has this symmetry, not its Laurent expansion $\mathrm{Z}^{\mathrm{PT}}_\beta(q)$ about $q = 0$; see example 1.2.2.

This result has been proven by T. Bridgeland in [Bri11], crucially using a structure result of the stable pair generating series of Y. Toda [Tod10a]. See section 2.5 for a discussion and analysis of these results.

1.2.4 DT invariants of birational Calabi–Yau threefolds

By a result of J. Kollár [Kol89], any two smooth projective Calabi–Yau threefolds are linked by a finite sequence of flops, since they are minimal models in the sense of the MMP. Let

$$\begin{tikzcd} \mathrm{Y} \arrow[dr, "f"'] & & \mathrm{Y}^+ \arrow[dl, "f^+"] \\ & \mathrm{X} & \end{tikzcd} \tag{1.2.20}$$

be a *flop* of Calabi–Yau threefolds. This means that f is a proper birational morphism with the property that if D is a divisor on Y such that $-\mathrm{D}$ is f-nef, then its proper transform D^+ on Y^+ is f^+-nef; the same property holds for the pair (f^+, Y^+). The morphisms f, f^+ only contract trees of rational curves to a point, so that X has rational singularities and a finite length singular locus.

In [Cal16a], J. Calabrese has shown a relation between the DT invariants on both sides of the flop. He introduced the *exceptional* generating series

$$\mathrm{DT}_{\mathrm{exc}}(\mathrm{Y})(z, q) \coloneqq \sum_{\substack{\beta \geq 0 \\ f_*\beta = 0}} \sum_{n \in \mathbf{Z}} \mathrm{DT}_n(\mathrm{Y}, \beta) q^n z^\beta \tag{1.2.21}$$

where q, z are formal variables, and where $f_*\beta = 0$ means that curves embedded in class β are supported on the schematic fibres of f. Set $\mathrm{DT}^\vee_{\mathrm{exc}}(\mathrm{Y})(z,q) = \mathrm{DT}_{\mathrm{exc}}(\mathrm{Y})(z^{-1},q)$.

Theorem 1.2.12. [Cal16a, Thm. 3.36] In the situation of diagram 1.2.20, we have

$$\mathrm{DT}^\vee_{\mathrm{exc}}(\mathrm{Y}) \cdot \mathrm{DT}(\mathrm{Y}) = \mathrm{DT}^\vee_{\mathrm{exc}}(\mathrm{Y}^+) \cdot \mathrm{DT}(\mathrm{Y}^+) \tag{1.2.22}$$

as an equality of generating series.

The proof uses T. Bridgeland's result [Bri02] that states that two birational Calabi–Yau threefolds have naturally equivalent derived categories, and that this equivalence sends the structure sheaf to the structure sheaf. In particular, there is a preferred derived equivalence

$$\Phi\colon \mathrm{D}(\mathrm{Y}) \xrightarrow{\sim} \mathrm{D}(\mathrm{Y}^+) \quad \text{such that} \quad \Phi(\mathcal{O}_\mathrm{Y}) = \mathcal{O}_{\mathrm{Y}^+}. \tag{1.2.23}$$

Earlier, in [Tod13], Y. Toda had given a conditional proof of Theorem 1.2.12 by using M. Van den Bergh's theory of non-commutative crepant resolutions of [VdB04]. Again, the result was conditional on a certain structural property of the moduli stack of objects in the heart of any bounded t-structure on the bounded derived category of a smooth projective Calabi–Yau threefold; the issue was resolved in [Tod16a].

Restricting equation (1.2.22) to the exceptional curve classes yields the formula

$$\mathrm{DT}^\vee_{\mathrm{exc}}(\mathrm{Y}) \cdot \mathrm{DT}_{\mathrm{exc}}(\mathrm{Y}) = \mathrm{DT}^\vee_{\mathrm{exc}}(\mathrm{Y}^+) \cdot \mathrm{DT}_{\mathrm{exc}}(\mathrm{Y}^+), \tag{1.2.24}$$

where we note that the sum of any two exceptional curve classes is again exceptional. In particular, by dividing (1.2.22) by (1.2.24), this allows one to rewrite the former as

$$\frac{\mathrm{DT}(\mathrm{Y})}{\mathrm{DT}_{\mathrm{exc}}(\mathrm{Y})} = \frac{\mathrm{DT}(\mathrm{Y}^+)}{\mathrm{DT}_{\mathrm{exc}}(\mathrm{Y}^+)}, \tag{1.2.25}$$

a formula also obtained in [Tod13, Thm. 5.8]. This quotient appears in the statement of the crepant resolution conjecture. Its terms can be given a geometrical interpretation, by generalising the notion of stable pair, that is crucial in our proof of the conjecture.

1.2.5 Generalisations

There are a number of ways in which the above situations can be generalised: one can either change the space on which curves are counted, or change the counting theory itself. We mention two such generalisations, both of which play a role in this thesis.

The first consists in replacing the smooth projective Calabi–Yau threefold Y with an orbifold $\mathcal{X}$ satisfying the Calabi–Yau properties of definition 1.1.1.

Definition 1.2.13. An *orbifold* $\mathcal{X}$ is a smooth Deligne–Mumford stack of finite type (over $\mathbf{C}$) that has generically trivial stabilizers.

By the Keel–Mori theorem [KM97], every such orbifold admits a *coarse moduli space* $\pi\colon \mathcal{X} \to \mathrm{X}$ where the natural morphism $\mathcal{O}_\mathrm{X} \to \pi_*(\mathcal{O}_\mathcal{X})$ is an isomorphism. Throughout, we assume this coarse moduli space to be projective. Note that X is CY3 when $\mathcal{X}$ is.

In a similar vein, one can define Donaldson–Thomas invariants of $\mathcal{X}$, using the 'Hilbert schemes' of $\mathcal{X}$ defined as Quot functors of $\mathcal{O}_\mathcal{X}$ by M. Olsson and J. Starr [OS03].

The second generalisation is a curve counting theory introduced by J. Bryan and D. Steinberg in [BS16], a relative version of stable pair invariants, with the aim of studying the crepant resolution conjecture. This theory is dependent on

- the choice of a crepant resolution of singularities $f\colon \mathrm{Y} \to \mathrm{X}$
- with fibres of dimension at most one
- of a Gorenstein Calabi–Yau threefold X with rational singularities.

By [Kov00], the latter condition is equivalent to $\mathbf{R}f_*\mathcal{O}_\mathrm{Y} = \mathcal{O}_\mathrm{X}$.

Definition 1.2.14. A *crepant* resolution of singularities $f\colon \mathrm{Y} \to \mathrm{X}$ is a proper birational morphism such that $f^*\omega_\mathrm{X} = \omega_\mathrm{Y}$.

Examples of such resolutions are given by the minimal resolutions of Du Val surface singularities. Note that if X is a Calabi–Yau threefold, then so is Y.

Definition 1.2.15. Let $f\colon \mathrm{Y} \to \mathrm{X}$ be a crepant resolution as above. An *f-stable pair* or *Bryan–Steinberg pair* (G, s) consists of a one-dimensional sheaf G on Y and a section $s \in \mathrm{H}^0(\mathrm{Y}, \mathrm{G})$. This data satisfies two stability requirements:

(i) $\mathrm{coker}(s)$ pushes down to a zero-dimensional sheaf, i.e., $\mathrm{coker}(s) \in \mathtt{T}_f$, and

(ii) G admits no maps from such sheaves, i.e., $\mathrm{Hom}(\mathtt{T}_f, \mathrm{G}) = 0$,

where $\mathtt{T}_f \coloneqq \{\mathrm{T} \in \mathrm{Coh}_{\leq 1}(\mathrm{Y}) \mid \mathbf{R}f_*(\mathrm{T}) \in \mathrm{Coh}_0(\mathrm{X})\}$.

Note that if f is the identity morphism, this notion reduces to that of stable pair in the sense of Pandharipande–Thomas. The authors of [BS16] construct invariants counting f-stable pairs of a fixed curve class, all of which we collect in a generating series $\mathrm{PT}_f(\mathrm{Y}/\mathrm{X})$ as per usual. They prove the following comparison result, providing a geometric interpretation of the coefficients in J. Calabrese's flop formula (1.2.22).

Theorem 1.2.16. [BS16, Thm. 6] Let $f\colon \mathrm{Y} \to \mathrm{X}$ be a small crepant resolution of the Calabi–Yau threefold X such that $\mathbf{R}f_*\mathcal{O}_\mathrm{Y} = \mathcal{O}_\mathrm{X}$. There is an equality of Laurent series

$$\mathrm{PT}_f(\mathrm{Y}/\mathrm{X}) = \frac{\mathrm{DT}(\mathrm{Y})}{\mathrm{DT}_{\mathrm{exc}}(\mathrm{Y})}. \tag{1.2.26}$$

1.3 The crepant resolution conjecture

The main topic of this thesis is the crepant resolution conjecture for Donaldson–Thomas invariants, as stated by J. Bryan, C. Cadman, and B. Young in [BCY12]. We give a reinterpretation of this conjecture as an equality of rational functions that is strictly necessary, and prove it via wall-crossing methods and the motivic Hall algebra.

After describing the setting of the crepant resolution conjecture we briefly describe its origins and previous work towards its proof.

1.3.1 Statement of the conjecture

The crepant resolution conjecture relates the Donaldson–Thomas generating series of a certain type of three-dimensional Calabi–Yau orbifold $\mathcal{X}$ to that of a particular crepant resolution Y of its coarse moduli space X. We have the following diagram.

$$\begin{tikzcd} \mathcal{X} \arrow[dr, "\pi"'] & & \mathrm{Y} \arrow[dl, "f"] \\ & \mathrm{X} & \end{tikzcd} \tag{1.3.1}$$

Here $\pi\colon \mathcal{X} \to \mathrm{X}$ is a finite morphism and f is the crepant resolution of singularities given by a certain Hilbert scheme of non-stacky points on $\mathcal{X}$. This is a global version of the McKay correspondence of [BKR01]; see section 2.4 for full details.

As a part of this correspondence, there is an equivalence of derived categories

$$\Phi\colon \mathrm{D}(\mathrm{Y}) \longrightarrow \mathrm{D}(\mathcal{X}) \tag{1.3.2}$$

that sends $\Phi(\mathcal{O}_{\mathrm{Y}}) = \mathcal{O}_{\mathcal{X}}$. Since Donaldson–Thomas invariants count quotients of the structure sheaf in the abelian category of coherent sheaves, the fact that Φ identifies the structure sheaves of Y and $\mathcal{X}$ is crucial for this equivalence to be of use in proving a relation between Donaldson–Thomas invariants.

We impose an additional restriction on the class of orbifolds under consideration, namely that they satisfy the *hard Lefschetz* condition. This condition is equivalent to the requirement that the fibres of the resolution f be at most one-dimensional, i.e., $\dim f^{-1}(x) \leq 1$ for all $x \in \mathrm{X}$; see Definition 2.4.13.

How does the equivalence of 1.3.2 allow us to identify curves on $\mathcal{X}$ and Y? Note that Φ induces a linear isomorphism on the level of numerical K-groups (see section 2.1.1), which we denote by $\phi\colon \mathrm{N}(\mathrm{Y}) \to \mathrm{N}(\mathcal{X})$. There is a natural filtration of these finite rank free abelian groups by the dimension of the support of sheaves. We write

$$\mathrm{N}_{\leq i}(\mathrm{Y}) := \langle [\mathrm{F}] \in \mathrm{N}(\mathrm{Y}) \mid \mathrm{F} \in \mathrm{Coh}(\mathrm{Y}) \text{ s.t.} \dim \mathrm{supp}(\mathrm{F}) \leq i \rangle, \tag{1.3.3}$$

and we loosely refer to elements of $\mathrm{N}_0(\mathrm{Y})$ and $\mathrm{N}_{\le 1}(\mathrm{Y})$ as point classes and curve classes respectively; we use the same terminology for the analogous groups on $\mathcal{X}$. However, the equivalence Φ is not compatible with these filtrations. There are a number of natural subgroups, and we summarize this numerical setup in a diagram.

$$\begin{tikzcd}
\mathrm{N}_0(\mathrm{Y}) \arrow[r, hook] & \mathrm{N}_{\mathrm{exc}}(\mathrm{Y}) \arrow[r, hook] \arrow[d, equal, "\phi"] & \mathrm{N}_{\le 1}(\mathrm{Y}) \arrow[d, equal, "\phi"] & \\
& \mathrm{N}_0(\mathcal{X}) \arrow[r, hook] & \mathrm{N}_{\mathrm{mr}}(\mathcal{X}) \arrow[r, hook] & \mathrm{N}_{\le 1}(\mathcal{X})
\end{tikzcd} \tag{1.3.4}$$

Here we define the *exceptional* classes on Y via $\mathrm{N}_{\mathrm{exc}}(\mathrm{Y}) \coloneqq \phi^{-1}(\mathrm{N}_0(\mathcal{X}))$ and the *multi-regular* classes via $\mathrm{N}_{\mathrm{mr}}(\mathcal{X}) \coloneqq \phi(\mathrm{N}_{\le 1}(\mathrm{Y}))$. By the hard Lefschetz condition, the former are classes supported on the one-dimensional fibres of f. The latter are those classes on $\mathcal{X}$ that correspond to curve classes on Y; the etymology of multi-regular has to do with the representation type of the class. These claims are justified by Lemma 2.4.17.

We introduce formal variables $\{t^\alpha \mid \alpha \in \mathrm{N}(\mathcal{X})\}$ to bookkeep Donaldson–Thomas invariants on $\mathcal{X}$ of class α, and use the identification $\phi\colon \mathrm{N}(\mathrm{Y}) \to \mathrm{N}(\mathcal{X})$ for those on Y. We define the generating series

$$\begin{aligned}
\mathrm{DT}(\mathcal{X})_\beta &\coloneqq \sum_{c \in \mathrm{N}_0(\mathcal{X})} \mathrm{DT}_{\mathcal{X}}(\beta + c) t^{\beta + c} \\
\mathrm{DT}(\mathcal{X})_0 &\coloneqq \sum_{c \in \mathrm{N}_0(\mathcal{X})} \mathrm{DT}_{\mathcal{X}}(c) t^{c}
\end{aligned} \tag{1.3.5}$$

of Donaldson–Thomas invariants on the orbifold $\mathcal{X}$, and the generating series

$$\begin{aligned}
\mathrm{DT}(\mathrm{Y})_\beta &\coloneqq \sum_{c \in \mathrm{N}_0(\mathcal{X})} \mathrm{DT}_{\mathrm{Y}}(\beta + c) t^{\beta + c} \\
\mathrm{DT}(\mathrm{Y})_{\mathrm{exc}} &\coloneqq \sum_{c \in \mathrm{N}_0(\mathcal{X})} \mathrm{DT}_{\mathrm{Y}}(c) t^{c}
\end{aligned} \tag{1.3.6}$$

of Donaldson–Thomas invariants on the resolution Y; recall that $\phi(\mathrm{N}_{\mathrm{exc}}(\mathrm{Y})) = \mathrm{N}_0(\mathcal{X})$.

Remark 1.3.1. On Y the holomorphic Euler characteristic $\chi(\mathrm{Y}, -)\colon \mathrm{N}_{\le 1}(\mathrm{Y}) \to \mathrm{N}_0(\mathrm{Y})$ defines a canonical splitting of the natural inclusion $i\colon \mathrm{N}_0(\mathrm{Y}) \hookrightarrow \mathrm{N}_{\le 1}(\mathrm{Y})$. Hence any class $\alpha \in \mathrm{N}_{\le 1}(\mathrm{Y})$ can be written uniquely as $(n_\alpha, \delta_\alpha) \in \mathrm{N}_0(\mathrm{Y}) \oplus \mathrm{N}_1(\mathrm{Y})$ with $\chi(\mathrm{Y}, \delta_\alpha) = 0$, where we write $\mathrm{N}_1(\mathrm{Y}) \coloneqq \mathrm{N}_{\le 1}(\mathrm{Y})/\mathrm{N}_0(\mathrm{Y})$ for the filtration quotient. In turn, we may define variables $t^\alpha = z^{\delta_\alpha} q^{n_\alpha}$ separately keeping track of the curve and point classes.

On the orbifold, however, $\mathrm{rk}\,\mathrm{N}_0(\mathcal{X}) > 1$ and there need not be such a canonical splitting available. We may of course *choose* a splitting $s\colon \mathrm{N}_{\le 1}(\mathcal{X}) \to \mathrm{N}_0(\mathcal{X})$ of the natural inclusion $i\colon \mathrm{N}_0(\mathcal{X}) \hookrightarrow \mathrm{N}_{\le 1}(\mathcal{X})$ to arrive at a presentation of the generating series in terms of curve class and point class variables, but we cannot guarantee any properties.

The original crepant resolution conjecture of [BCY12] states the following.

Conjecture 1.3.2. [BCY12, Conj. 1] Let $\mathcal{X}$ be a smooth CY3 orbifold with projective coarse moduli space satisfying the hard Lefschetz condition, let $\beta \in \mathrm{N}_{1,\mathrm{mr}}(\mathcal{X})$ be a multi-regular curve class. Then

$$\frac{\mathrm{DT}(\mathcal{X})_\beta}{\mathrm{DT}(\mathcal{X})_0} = \frac{\mathrm{DT}(\mathrm{Y})_\beta}{\mathrm{DT}_{\mathrm{exc}}(\mathrm{Y})} \tag{1.3.7}$$

are equal as generating series upon identifying formal variables via ϕ.

In chapter 3, we construct a counterexample disproving equation (1.3.7) in general. An equality is obtained in this example, however, when both sides of the equation are interpreted as the Laurent expansions at *different* point of the *same* multi-variable rational function. This suggests a 'corrected' version of Conjecture 1.3.2.

Remark 1.3.3. In [BCY12, Conj. 2] a second result is conjectured, relating the zero-dimensional invariants of $\mathcal{X}$ to the exceptional invariants of Y. To be precise, the authors conjecture the equality of Laurent series

$$\mathrm{DT}(\mathcal{X})_0 = \frac{\mathrm{DT}_{\mathrm{exc}}(\mathrm{Y})\mathrm{DT}^\vee_{\mathrm{exc}}(\mathrm{Y})}{\mathrm{DT}_0(\mathrm{Y})}, \tag{1.3.8}$$

where $\mathrm{DT}^\vee_{\mathrm{exc}}(\mathrm{Y})(z,q) = \mathrm{DT}_{\mathrm{exc}}(\mathrm{Y})(z^{-1},q)$ as a function of the formal variables z, q. This relation has been proved by J. Calabrese in [Cal16b] and requires no rationality.

We are now in a position to state our reinterpretation of the crepant resolution conjecture for Donaldson–Thomas invariants.

Theorem (Theorem 5.1.1)**.** Let $\mathcal{X}$ be a smooth CY3 orbifold with projective coarse moduli space satisfying the hard Lefschetz condition. For each multi-regular curve class $\beta \in \mathrm{N}_{\mathrm{mr}}(\mathcal{X})$ there exists a rational function $f_\beta(q)$, where $q = (q_1, q_2, \ldots, q_r)$ are generators of $\mathbf{Q}[\mathrm{N}_0(\mathcal{X})]$ corresponding to a basis of $\mathrm{N}_0(\mathcal{X})$, such that

1. a certain expansion of $f_\beta(q)$ is the quotient $\mathrm{DT}(\mathcal{X})_\beta/\mathrm{DT}(\mathcal{X})_0$ of formal power series, another is the quotient $\mathrm{DT}(\mathrm{Y})_\beta/\mathrm{DT}_{\mathrm{exc}}(\mathrm{Y})$ of formal power series, and
2. we may write $f_\beta(q) = g/h$ with $g, h \in \mathbf{Z}[\mathrm{N}_0(\mathcal{X})]$ in such a way that h is of the form $h = (1 - q^{2\beta\cdot\mathrm{A}})^n$ for some ample divisor A on X and some positive integer n.

Remark 1.3.4. See Theorem 5.1.1 for the precise meaning of expansion in the multivariate case.

Remark 1.3.5. Unfortunately, the above bound on the order of the poles is far from sharp. For example, take for $\beta = \mathrm{L}$ the class of a line on the (smooth projective) quintic threefold $\mathcal{X}$, let $\mathrm{PT}(\mathcal{X})_{\mathrm{L}}$ denote the generating series of PT invariants, and let $f_{\mathrm{L}}(q)$ denote its rational re-summation of 1.2.9. Then [BB07, Cor. 3.3] shows that $(1+q)^2 f_{\mathrm{L}}(q) = 2875q$ is polynomial, so its poles lie at ± 1. However, the above theorem only shows that $(1+q^5)^2 f_{\mathrm{L}}(q)$ is polynomial, i.e., its poles lie at tenth roots of unity.

Remark 1.3.6. Drawing inspiration from the DT/PT correspondence for varieties 1.2.9, the quotient $\mathrm{DT}(\mathcal{X})_\beta/\mathrm{DT}(\mathcal{X})_0$ should have an interpretation as the generating function of stable pair invariants on $\mathcal{X}$. Moreover, by 1.2.11, it should be the expansion of a rational function. Similarly, Theorem 1.2.16 relates the quotient $\mathrm{DT}(\mathrm{Y})_\beta/\mathrm{DT}_{\mathrm{exc}}(\mathrm{Y})$ to *relative* stable pair invariants, the generating function of which should be the expansion of a rational function too.

1.3.2 On the proof

Our strategy to prove Theorem 5.1.1 is the following:

1. We follow the approach initiated by Bridgeland for his proof of the DT/PT correspondence in [Bri11], and later used by various authors [Cal16a, BS16, Tod16a, Obe16], encoding a change of stability condition in the motivic Hall algebra.

2. The general strategy is to prove a categorical identity in a Hall algebra H relating stable pairs on $\mathcal{X}$ to Bryan–Steinberg pairs on Y/X; the relation between sheaves on $\mathcal{X}$ and Y is provided by the McKay equivalence $\Phi\colon \mathrm{D}(\mathrm{Y}) \cong \mathrm{D}(\mathcal{X})$.

3. Crucially, H is *not* the Hall algebra of $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ used by T. Bridgeland, but the Hall algebra of a certain abelian category of two-term complexes in $\mathrm{D}(\mathcal{X})$. Thus we study curves via their ideal sheaves, as opposed to their structure sheaf with a section, allowing us to put embedded curves and pairs on an equal footing.

4. We apply the integration morphism from H to a power series ring, taking Behrend weighted Euler characteristics, obtaining a relation between their generating series.

Unfortunately, in practice, this strategy does not work since the difference between the two notions of curve is ‘too large’. We solve this in the following way:

1. We define a notion of curve dependent on the choice of a torsion pair in $\mathrm{Coh}_{\leq 1}(\mathcal{X})$, and prove a universal wall-crossing formulate relating these notions in H.

2. We break up the formula between stable pairs on $\mathcal{X}$ and Bryan–Steinberg pairs on Y/X into infinitely many smaller formulas that are ‘integrable’.

3. We show that, for a fixed curve class, these infinitely many formulas organise into finitely many clusters, each clusters representing infinitely many wall-crossings.

4. We prove that the stable pair generating function is rational.

5. We prove that after integration, on the level of generating series, crossing a cluster corresponds to re-expanding the rational function.

6. We identify the re-expansion after the final cluster as Bryan–Steinberg invariants.

This proof is interesting in a number of ways:

1. It is the first known case of wall-crossing of surface classes (on Y), resulting in an Euler pairing that depends on the 'crossing' wall *and* the 'crossed' one.

2. The interpretation of a cluster of infinitely many wall-crossings as a re-expansion of a rational function seems new, opening up this strategy to other problems.

3. Recognising the re-expansion of a rational function by its expansions alone.

1.3.3 Origins and previous work

Donaldson–Thomas invariants are considered to be the mathematical counterparts of BPS state counts in topological string theory compactified on X. Principles of physics indicate that the string theory of an orbifold Calabi–Yau threefold and that of any of its crepant resolutions ought to be equivalent. Thus one expects the DT theories of an orbifold and its crepant resolutions to be equivalent in some way.

This relation was first written down by Y. Ruan for cohomology in [Rua06]. There is a Gromov–Witten version in the hard Lefschetz case by J. Bryan and T. Graber [BG09b], and a more general statement by T. Coates and Y. Ruan [CR13].

The statement of the hard Lefschetz crepant resolution conjecture for Donaldson–Thomas invariants has been proven for point classes and fibre classes by J. Calabrese in [Cal16b]; no rationality is required, the statement is an equality of generating series. A proof of the general case of toric CY3 orbifolds with transverse A_n-singularities was claimed by D. Ross in [Ros17], but we provide a counterexample to this statement in Chapter 3.

1.4 Results of this thesis

We describe the results of this thesis after dividing them into two sets.

The first set of results is about *pairs*, a generalisation of the notion of a curve on a smooth projective Calabi–Yau threefold Y. A pair is associated to a torsion pair on $\mathrm{Coh}_{\leq 1}(\mathrm{Y})$, which we think of as a rough notion of stability. Examples of pairs are ideal sheaves of curves and stable pairs in the sense of [PT10], thus putting these notions on an equal footing. We prove basic results about pairs, such as conditions for which their moduli stack exists and is a $\mathbf{C}^*$-gerbe. We obtain pair counting invariants by taking the Behrend weighted Euler characteristic. Moreover, we establish a universal wall-crossing formula in a motivic Hall algebra relating all notions of pairs and hence, upon applying the integration map, the associated counting invariants.

The second set of results is joint work with J. Calabrese and J. Rennemo. It constitutes a full proof of the crepant resolution conjecture for Donaldson–Thomas invariants as stated in Theorem 5.1.1. Results of the first set are used in a crucial way.

Finally, we indicate a number of interesting corollaries that follow from our proof. The main corollary is a general rationality result for the generating series of stable pair invariants on any smooth three-dimensional Calabi–Yau orbifold.

1.4.1 Pairs and their wall-crossing

Fix a smooth three-dimensional Calabi–Yau orbifold $\mathcal{X}$; in particular, $\mathcal{X}$ could be a smooth projective variety. Recall that this means that $\omega_{\mathcal{X}} \cong \mathcal{O}_{\mathcal{X}}$ and $\mathrm{H}^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = 0$.

Sheaf-theoretic notions of a *curve* in $\mathcal{X}$ are typically defined as a pair (F, s), consisting of a one-dimensional sheaf F and a section $s\colon \mathcal{O}_{\mathcal{X}} \to \mathrm{F}$, that satisfies certain conditions. In [Tod10a], Y. Toda introduces the subcategory

$$\mathtt{A} := \langle \mathcal{O}_{\mathcal{X}}[1], \mathrm{Coh}_{\leq 1}(\mathcal{X}) \rangle_{\mathrm{ex}} \subset \mathrm{D}^{[-1,0]}(\mathcal{X}) \tag{1.4.1}$$

as the extension-closure of $\mathcal{O}_{\mathcal{X}}[1]$ and $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ in the bounded derived category $\mathrm{D}^b(\mathcal{X})$. Furthermore, he proves that it is a noetherian abelian category; see section 4.1.1.

We introduce a notion of curve object on $\mathcal{X}$ dependent on a choice of torsion pair.

Definition 1.4.1. Let $(\mathtt{T}, \mathtt{F})$ be a torsion pair on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$. A $(\mathtt{T}, \mathtt{F})$-*pair* is an object $\mathrm{E} \in \mathtt{A}$ of $\mathrm{rk}(\mathrm{E}) = -1$ such that

1. $\mathrm{Hom}(\mathrm{T}, \mathrm{E}) = 0$ for all $\mathrm{T} \in \mathtt{T}$,
2. $\mathrm{Hom}(\mathrm{E}, \mathrm{F}) = 0$ for all $\mathrm{F} \in \mathtt{F}$.

Remark 1.4.2. When no confusion is likely to arise, we refer to such objects simply as *pairs*. We write $\mathtt{Pair}(\mathtt{T}, \mathtt{F}) \subset \mathtt{A}$ for the corresponding subcategory.

Under a cohomological criterion on $\mathtt{T}$, all pairs are of a standard form.

Lemma 1.4.3. Let $(\mathtt{T},\mathtt{F})$ be a torsion pair on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ such that every $\mathrm{T} \in \mathtt{T}$ satisfies $\mathrm{H}^i(\mathcal{X},\mathrm{T}) = 0$ for all $i \neq 0$. Then an object $\mathrm{E} \in \mathtt{A}$ of rank -1 is a $(\mathtt{T},\mathtt{F})$-pair if and only if it is quasi-isomorphic to a two-term complex

$$\mathrm{E} = (\mathcal{O}_{\mathcal{X}} \xrightarrow{s} \mathrm{F}) \in \mathrm{D}^{[-1,0]}(\mathcal{X})$$

with $\mathrm{H}^0(\mathrm{E}) = \mathrm{coker}(s) \in \mathtt{T}$ and $\mathrm{F} \in \mathtt{F}$.

Example 1.4.4. In the case of the trivial torsion pair, a $(0, \mathrm{Coh}_{\leq 1}(\mathcal{X}))$-pair is just the ideal sheaf $\mathrm{I}_\mathrm{C}[1] \cong (\mathcal{O}_{\mathcal{X}} \twoheadrightarrow \mathcal{O}_\mathrm{C})$ of a curve $\mathrm{C} \subset \mathcal{X}$.

Moreover, stable pairs in the sense of Pandharipande–Thomas [PT09] are examples of pairs. Indeed, choosing the PT torsion pair, $\mathtt{T}_{\mathrm{PT}} = \mathrm{Coh}_0(\mathcal{X})$ and $\mathtt{F}_{\mathrm{PT}} = \mathrm{Coh}_1(\mathcal{X})$, we see that a $(\mathtt{T}_{\mathrm{PT}}, \mathtt{F}_{\mathrm{PT}})$-pair is the same thing as a stable pair by the above criterion.

Under some mild assumptions on the torsion pair, the stack of pairs is an algebraic stack that is locally of finite type. To state the result, let $\mathfrak{Mum}_{\mathcal{X}}$ denote M. Lieblich's mother of all moduli stacks [Lie06]. Roughly speaking, it parametrises complexes $\mathrm{E} \in \mathrm{D}(\mathcal{X})$ such that $\mathrm{Ext}^i(\mathrm{E},\mathrm{E}) < 0$, i.e., E is an object of the heart of a bounded t-structure on $\mathrm{D}(\mathcal{X})$. He moreover shows that $\mathfrak{Mum}_{\mathcal{X}}$ is an algebraic stack locally of finite type.

Given a subcategory $\mathtt{C} \subset \mathrm{D}(\mathcal{X})$ whose objects have vanishing negative self-extensions, our general convention is to denote $\underline{\mathtt{C}} \subset \mathfrak{Mum}_{\mathcal{X}}$ the corresponding substack. A torsion pair $(\mathtt{T},\mathtt{F})$ on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ is said to be *open* if the moduli stacks $\underline{\mathtt{T}}, \underline{\mathtt{F}}$ are open[5] in $\underline{\mathrm{Coh}}_{\mathcal{X}}$.

Proposition 1.4.5. Let $(\mathtt{T},\mathtt{F})$ be an open torsion pair on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$. Assume that $\mathrm{Coh}_0(\mathcal{X}) \subset \mathtt{T}$, so $\mathtt{F} \subset \mathrm{Coh}_1(\mathcal{X})$. The substack $\underline{\mathtt{Pair}}(\mathtt{T},\mathtt{F}) \subset \mathfrak{Mum}_{\mathcal{X}}$ parametrising $(\mathtt{T},\mathtt{F})$-pairs is open. In particular, it is an algebraic stack locally of finite type.

Under a further condition on the torsion pair, $\underline{\mathtt{Pair}}(\mathtt{T},\mathtt{F})$ is a $\mathbf{C}^*$-gerbe. We call a torsion pair *numerical* if the classes of $\mathrm{T} \in \mathtt{T}$ and $\mathrm{F} \in \mathtt{F}$ in the numerical Grothendieck group of $\mathcal{X}$ are equal, $[\mathrm{T}] = [\mathrm{F}]$ in $\mathrm{N}(\mathcal{X})$, if and only if $\mathrm{T} = 0 = \mathrm{F}$. This condition always holds for torsion pairs induced by weak stability conditions on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$.

Lemma 1.4.6. Let $(\mathtt{T},\mathtt{F})$ be a numerical torsion pair on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$. If E is a $(\mathtt{T},\mathtt{F})$-pair, then $\mathrm{Aut}(\mathrm{E}) = \mathbf{C}^*$. In particular, $\underline{\mathtt{Pair}}(\mathtt{T},\mathtt{F})$ is a $\mathbf{C}^*$-gerbe over its coarse space.

All notions of pair are related via a universal wall-crossing formula in a motivic Hall algebra. Indeed, let $\mathrm{H}_\infty(\mathtt{A})$ denote[6] the infinite-type Hall algebra of the category $\mathtt{A}$.

[5] The stacks $\underline{\mathrm{Coh}}_{\leq 1,\mathcal{X}} \subset \underline{\mathrm{Coh}}_{\mathcal{X}} \subset \mathfrak{Mum}_{\mathcal{X}}$ are all open substacks; see section 2.2.2

[6] In fact, we work in a larger Hall algebra, but we stick to $\mathrm{H}_\infty(\mathtt{A})$ for expository purposes.

Objects are symbols $[\mathrm{S} \to \underline{\mathtt{A}}]$ where S is an algebraic stack locally of finite type with affine geometric stabilizers. Let $(\mathtt{T}, \mathtt{F})$ be a torsion pair on $\mathrm{Coh}_{\le 1}(\mathcal{X})$, and denote by

$$\mathcal{P}(\mathtt{T}, \mathtt{F}) := [\underline{\mathtt{Pair}}(\mathtt{T}, \mathtt{F}) \subset \underline{\mathtt{A}}] \in \mathrm{H}_\infty(\underline{\mathtt{A}}) \tag{1.4.2}$$

the corresponding object in the Hall algebra. Furthermore, define the characteristic function of an open substack of $\underline{\mathtt{A}}$ as $\mathbf{1}_\mathtt{T} := [\underline{\mathtt{T}} \subset \underline{\mathtt{A}}]$ and $\mathbf{1}_\mathtt{F} := [\underline{\mathtt{F}} \subset \underline{\mathtt{A}}]$.

Proposition 1.4.7. Given two torsion pairs $(\mathtt{T}, \mathtt{F})$ and $(\mathtt{T}', \mathtt{F}')$ on $\mathrm{Coh}_{\le 1}(\mathcal{X})$. There is a universal wall-crossing formula

$$\mathbf{1}_\mathtt{T} * \mathcal{P}(\mathtt{T}, \mathtt{F}) * \mathbf{1}_\mathtt{F} = \mathbf{1}_{\mathtt{T}'} * \mathcal{P}(\mathtt{T}', \mathtt{F}') * \mathbf{1}_{\mathtt{F}'} \tag{1.4.3}$$

in the infinite-type motivic Hall algebra $\mathrm{H}_\infty(\mathtt{A})$.

If two torsion pairs are 'sufficiently close', the above identity holds in a finite-type Hall algebra and, as a consequence, can be integrated into an equality in the quantum torus. To be a bit more precise, let $(\mathtt{T}_-, \mathtt{F}_-)$ and $(\mathtt{T}_+, \mathtt{F}_+)$ denote two torsion pairs on $\mathrm{Coh}_{\le 1}(\mathcal{X})$, and assume without loss of generality that $\mathtt{T}_+ \subset \mathtt{T}_-$. We set $\mathtt{W} = \mathtt{T}_- \cap \mathtt{F}_+$, and think of this subcategory as the 'walls' being crossed to get from $(\mathtt{T}_-, \mathtt{F}_-)$-pairs to $(\mathtt{T}_+, \mathtt{F}_+)$-pairs. The two torsion pairs are 'sufficiently close' if the category of walls $\mathtt{W}$ is 'sufficiently small'; see Definition 4.3.10 for the precise statement.

Proposition 1.4.8. Let $(\mathtt{T}_-, \mathtt{F}_-)$ and $(\mathtt{T}_+, \mathtt{F}_+)$ denote two torsion pairs on $\mathrm{Coh}_{\le 1}(\mathcal{X})$ that are 'sufficiently close', then there is the wall-crossing formula

$$\mathbf{1}_\mathtt{W} * \mathcal{P}_- = \mathcal{P}_+ * \mathbf{1}_\mathtt{W}, \tag{1.4.4}$$

where $\mathcal{P}_\pm = [\underline{\mathtt{Pair}}(\mathtt{T}_\pm, \mathtt{F}_\pm) \subset \underline{\mathtt{A}}]$, in a certain finite-type Hall algebra $\mathrm{H}_{\mathrm{gr}}(\mathtt{A})$ of $\mathtt{A}$.

The *quantum torus* is a commutative Poisson algebra $\mathbf{Q}[\mathrm{N}(\mathcal{X})]$ generated as a vector space by symbols $\{t^\alpha \mid \alpha \in \mathrm{N}(\mathcal{X})\}$. The product and Poisson bracket are defined as

$$t^\alpha \cdot t^\beta = (-1)^{\chi(\alpha,\beta)} t^{\alpha+\beta} \quad \text{and} \quad \{t^\alpha, t^\beta\} = \chi(\alpha, \beta) t^\alpha \cdot t^\beta \tag{1.4.5}$$

respectively, where $\chi\colon \mathrm{N}(\mathcal{X}) \times \mathrm{N}(\mathcal{X}) \to \mathbf{Z}$ denotes the bilinear Euler pairing. Note that it is an antisymmetric pairing by the Calabi–Yau condition on $\mathcal{X}$ and Serre duality.

The stack $\underline{\mathtt{A}}$ decomposes as a disjoint union of open and closed substacks $\underline{\mathtt{A}} = \bigoplus_{\alpha \in \mathrm{N}(\mathcal{X})} \underline{\mathtt{A}}_\alpha$ where $\underline{\mathtt{A}}_\alpha$ parametrises objects in $\underline{\mathtt{A}}$ of class α. This induces a grading by $\mathrm{N}(\mathcal{X})$ on the Hall algebra compatible with the product structure, since taking an extension of an object of class α by an object of class β yields an object of class $\alpha + \beta$. Thus the Hall algebra is naturally an $\mathrm{N}(\mathcal{X})$-graded algebra.

Let I denote the integration morphism $\mathrm{I}: \mathrm{H}(\mathtt{A}) \to \mathbf{Q}[\mathrm{N}(\mathcal{X})]$. It is a graded Poisson algebra morphism that sends a symbol $[f: \mathrm{X} \to \underline{\mathtt{A}}]$ to the Behrend weighted Euler characteristic $e_{\mathrm{B}}(\mathrm{X})t^{\alpha} \in \mathrm{N}(\mathcal{X})$ if f factors through $\underline{\mathtt{A}}_{\alpha}$.

We have the following numerical wall-crossing formula between different pairs.

Theorem 1.4.9. Let $(\mathtt{T}_+, \mathtt{W}, \mathtt{F}_-)$ be an open numerical torsion triple on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$, and assume that it is wall-crossing material. Then $w := \mathrm{I}_{\mathrm{gr}}\left((\mathbf{L}-1)\log \mathbf{1}_{\mathtt{W}}\right)$ is well defined as a formal power series in $\mathbf{Q}\{\mathrm{N}(\mathcal{X})\}$, the abelian group of infinite formal sums of terms $a_c q^c$ with $c \in \mathrm{N}_0(\mathcal{X})$, where $\mathbf{L} = [\mathbf{A}^1_{\mathbf{C}}]$, and there is the identity

$$\mathrm{I}_{\mathrm{gr}}\left((\mathbf{L}-1)\mathcal{P}_+\right) = \exp\left(\{w, -\}\right)\mathrm{I}_{\mathrm{gr}}\left((\mathbf{L}-1)\mathcal{P}_-\right) \tag{1.4.6}$$

in $\mathbf{Q}\{\mathrm{N}(\mathcal{X})\}$, where $\mathcal{P}_{\pm} = [\underline{\mathtt{Pair}}(\mathtt{T}_{\pm}, \mathtt{F}_{\pm}) \subset \underline{\mathtt{C}}] \in \mathrm{H}_{\mathrm{gr}}(\mathtt{C})$ as before.

1.4.2 A proof of the crepant resolution conjecture

Let $\mathcal{X}$ be a smooth three-dimensional Calabi–Yau orbifold satisfying the hard Lefschetz condition, let $\pi: \mathcal{X} \to \mathrm{X}$ be its coarse moduli space, and let $f: \mathrm{Y} \to \mathrm{X}$ be the crepant resolution described in section 1.3. The results in this section are joint work with J. Calabrese and J. Rennemo.

Our proof of the crepant resolution conjecture Theorem 5.1.1 proceeds in three steps.

First, we interpret the left-hand side as the generating function of stable pair invariants on $\mathcal{X}$, thus proving an orbifold DT/PT correspondence.

Theorem 1.4.10. Let $\mathcal{X}$ be a smooth CY3 orbifold with projective coarse moduli space satisfying the hard Lefschetz condition, and let $\beta \in \mathrm{N}_{1,\mathrm{mr}}(\mathcal{X})$ be a multi-regular curve class. Then there is an equality of Laurent series

$$\mathrm{PT}(\mathcal{X})_{\beta} = \frac{\mathrm{DT}(\mathcal{X})_{\beta}}{\mathrm{DT}(\mathcal{X})_0}. \tag{1.4.7}$$

Second, we prove a general rationality statement about the stable pair theory of the orbifold, analogous to the theorem for varieties as in Conjecture 1.2.9. For this statement the hard Lefschetz condition is not necessary.

Theorem 1.4.11. Let $\mathcal{X}$ be a smooth CY3 orbifold with projective coarse moduli space, and let $\beta \in \mathrm{N}_{\leq 1}(\mathcal{X})$ be any curve class. Then $\mathrm{PT}(\mathcal{X})_{\beta}$ is the Laurent expansion of a rational function $f_{\beta}(q)$, where q denotes a multi-variable $q_1, q_2, \ldots, q_r$ and the q_i are generators of $\mathbf{Q}[\mathrm{N}_0(\mathcal{X})]$ corresponding to a basis of $\mathrm{N}_0(\mathcal{X})$.

And third, using the universal wall-crossing formula of the previous section, we prove that the generating series of Bryan–Steinberg invariants associated to the crepant resolution f is a different expansion of the rational function $f_{\beta}(q)$.

Theorem 1.4.12. Let $\mathcal{X}$ be a smooth CY3 orbifold with projective coarse moduli space satisfying the hard Lefschetz condition, let $\beta \in \mathrm{N}_{1,\mathrm{mr}}(\mathcal{X})$ be a multi-regular curve class. The generating function $\mathrm{BS}(\mathrm{Y}/\mathrm{X})_\beta$ is another expansions of the rational function $f_\beta(q)$.

Together with Theorem 1.2.16, this completes the proof of Theorem 5.1.1.

Remark 1.4.13. The condition that the coarse moduli space X be projective can be weakened to the condition that it be quasi-projective and that Pic(X) is a finitely generated abelian group; this will be discussed in the forthcoming [BCR], but remains conditional on the extension of the Behrend function identities of [Tod16a, Thm. 2.6] to the quasi-projective setting.

However, the topological analogue of this conjecture, i.e., the statement obtained by replacing the Behrend weighted e_{B} by the ordinary topological e, holds in general.

It is now clear why Theorem 5.1.1 is a question of types two through five as discussed in section 1.2: its proof requires a comparison of DT and PT curve counting invariants of $\mathcal{X}$ (type 2), it requires a comparison of PT invariants of $\mathcal{X}$ and relative PT invariants of Y (type 3), and these proofs make essential use of the rationality of the generating functions of counting invariants (type 4). Finally, note that this rationality is a generalisation (type 5) of known results (Theorem 1.2.9) in two ways:

1. the rationality of $\mathrm{PT}(\mathcal{X})_\beta$ to the case of smooth projective CY3 orbifolds,
2. the rationality of $\mathrm{BS}(\mathrm{Y}/\mathrm{X})_\beta$ to non-trivial crepant resolutions $f\colon \mathrm{Y} \to \mathrm{X}$ under the condition that β is a multi-regular curve class.

1.4.3 Corollaries to the proof

The most important ingredient of the proof might be the realisation that a well-known fact in the theory of generating functions has important applications in enumerative geometry: if the difference between the coefficients of two generating functions is quasi-polynomial, then the one generating function is an expansion of a rational function if and only if the other is, and moreover these functions are strongly related.

There are two main corollaries, one conjectural, to our proof of the crepant resolution conjecture, both of which are directly or indirectly proved via the above fact. The first concerns the rational function $f_\beta(q)$ of which $\mathrm{PT}(\mathcal{X})_\beta(q)$ is a Laurent expansion. It is a symmetry result analogous to the one for varieties in Theorem 1.2.11.

It should be possible to prove the statement for orbifolds with the methods of Chapter 5. Since we have not completed this proof so far, we include it as a conjecture.

Conjecture 1.4.14. Let $\mathcal{X}$ be a smooth CY3 orbifold with projective coarse moduli space, and let $\beta \in \mathrm{N}_{\leq 1}(\mathcal{X})$ be a curve class. The rational function $f_\beta(q)$ has the symmetry

$$f_\beta(q) = f_{\beta^\vee}(q^\vee), \tag{1.4.8}$$

where $(-)^\vee \colon \mathbf{Q}[\mathrm{N}(\mathcal{X})] \to \mathbf{Q}[\mathrm{N}(\mathcal{X})]$ is the linear anti-isomorphism induced by the shifted derived dualising functor $\mathbf{D} = \mathbf{R}\,\underline{\mathrm{Hom}}(-, \mathcal{O}_\mathcal{X})[2] \colon \mathrm{D}(\mathcal{X}) \to \mathrm{D}(\mathcal{X})$.

Note that $\mathbf{D}(\mathrm{Coh}_0(\mathcal{X})) = \mathrm{Coh}_0(\mathcal{X})[-1]$ and $\mathbf{D}(\mathrm{Coh}_1(\mathcal{X})) = \mathrm{Coh}_1(\mathcal{X})$ imply that the dualising isomorphism preserves the filtration by dimension of support. In other words, $q^\vee \in \mathbf{Q}[\mathrm{N}_0(\mathcal{X})]$ and $\beta^\vee \in \mathbf{Q}[\mathrm{N}_{\leq 1}(\mathcal{X})]$ as required.

Remark 1.4.15. Contrary to the dualising functor for smooth projective varieties, the dualising functor for orbifolds does not fix all curve classes. Indeed, the dualisation also affects the representation-type of a class. For example, $\mathbf{D}(\mathcal{O}_\mathrm{C} \otimes \rho) = \mathcal{O}_\mathrm{C} \otimes \rho^\vee$ where $\mathrm{C} \subset \mathcal{X}$ is a Cohen–Macaulay curve in the stacky locus of $\mathcal{X}$, whose generic point has non-trivial representation type ρ, and $\rho^\vee$ denotes the dual representation of ρ.

Note, however, that if β is multi-regular then $\beta^\vee = \beta + c_\beta$ for a certain $c_\beta \in \mathrm{N}_0(\mathcal{X})$.

The second corollary is a direct consequence of the crepant resolution conjecture, and it concerns a rationality and symmetry statement for relative stable pairs. Note that the symmetry statement is conditional on Conjecture 1.4.14 above.

Theorem 1.4.16. Let $\mathcal{X}$ be a smooth CY3 orbifold satisfying the hard Lefschetz condition with projective coarse space, and let $f \colon \mathrm{Y} \to \mathrm{X}$ be the natural crepant resolution.

1. For $\beta \in \mathrm{N}_{\mathrm{mr}}(\mathcal{X})$, the generating function of relative stable pair invariants $\mathrm{BS}_f(\mathrm{Y}/\mathrm{X})_\beta$, is the Laurent expansion of a rational function $\mathrm{R}^f_\beta(q)$.
2. This rational function is invariant under the symmetry $(-)^\vee$, that is

$$\mathrm{R}^f_\beta(q) = \mathrm{R}^f_{\beta^\vee}(q^\vee). \tag{1.4.9}$$

Recall that $\beta^\vee = \beta + c_\beta$ for some $c_\beta \in \mathrm{N}_0(\mathcal{X})$ for any multi-regular curve class β on $\mathcal{X}$.

Finally, by J. Calabrese's flop formula (1.2.22), these two results follow for *any* crepant resolution of singularities, not just the natural one of [BKR01]. Again, the symmetry statement is conditional on Conjecture 1.4.14 above.

Theorem 1.4.17. Let $\mathcal{X}$ be a smooth CY3 orbifold satisfying the hard Lefschetz condition with projective coarse space, and let $g \colon \mathrm{Z} \to \mathrm{X}$ be any crepant resolution. The results of the previous theorem hold for $\mathrm{BS}_g(\mathrm{Z}/\mathrm{X})_\beta$ and $\mathrm{R}^g_\beta(q)$.

More informally, we conclude that *defining* the curve-counting invariants (be they DT or PT) of the singular Gorenstein Calabi–Yau threefold X via its stacky resolution $\mathcal{X}$ or via any crepant resolution $g\colon \mathrm{Z} \to \mathrm{X}$, the result is the same.

1.5 Future directions

There are a number of questions related to the material of this thesis that we feel would be interesting to pursue in future work. In the following, these questions are loosely organised based on whether the nature of the counts is numerical, motivic, or categorical.

1.5.1 Classical curve counting

Implications for the GW crepant transformation conjecture

By taking the crepant resolution conjecture through the GW/PT correspondence, we obtain a crepant resolution conjecture for Gromov–Witten invariants. However, the GW/PT correspondence is, as far as we know, not yet established for orbifolds.

Higher rank crepant resolution conjecture

Let Y be a smooth projective Calabi–Yau threefold with ample class ω, let $r \in \mathbf{Z}_{\geq 1}$, let $\mathrm{D} \in \mathrm{H}^2(\mathrm{Y}, \mathbf{Z})$ be a divisor class such that $\gcd(r, \mathrm{D} \cdot \omega^2) = 1$, and set $\mu = \mathrm{D} \cdot \omega^2 / r \in \mathbf{Q}$.

In [Tod16a], Y. Toda proves a higher rank analogue of the DT/PT correspondence. Here DT objects are ω-slope stable sheaves of class (r, D), a natural generalisation of ideal sheaves that are slope stable of class $(1, 0)$, and PT objects are J. Lo's higher rank PT-stable objects [Lo12]. Note that the coprimality condition guarantees that any ω-slope semistable sheaf of class (r, D) is ω-slope stable, whence the corresponding moduli stack of such is a $\mathbf{C}^*$-gerbe over its coarse moduli space.

Y. Toda's proof of this result is a generalisation of his proof of the DT/PT correspondence. It proceeds by various wall-crossings, described via nested torsion pairs, in the noetherian abelian category

$$\mathtt{A}_\mu^{\mathrm{Y}} = \langle \mathrm{Coh}_\mu(\mathrm{Y})[1], \mathrm{Coh}_{\leq 1}(\mathrm{Y}) \rangle_{\mathrm{ex}} \subset \mathrm{D}^{[-1,0]}(\mathrm{Y}), \tag{1.5.1}$$

where $\mathrm{Coh}_\mu(\mathrm{Y}) \subset \mathrm{Coh}(\mathrm{Y})$ denotes the abelian subcategory generated by ω-slope semistable sheaves of slope μ and the zero sheaf; setting $\mu = 0$ recovers $\mathtt{A}$.

Let $\mathcal{X}$ be a smooth CY3 orbifold with projective coarse moduli space, and let $\mathtt{A}_\mu$ denote the analogous category. Given a torsion pair $(\mathtt{T}, \mathtt{F})$ on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ we obtain a notion of higher rank $(\mathtt{T}, \mathtt{F})$-pair by requiring $\mathrm{rk}(\mathrm{E}) = -r$ in Definition 4.1.14. Again, it generalizes the notion of DT and PT objects. If the torsion pair is numerical, we find $\mathrm{Aut}(E) = \mathbf{C}^*$ for such a pair by the coprimality condition on (r, D).

The construction of the corresponding moduli stacks go through, provided $(\mathtt{T}, \mathtt{F})$ is *open*, and the universal wall-crossing formula holds in a modified Hall algebra. It is reasonable to expect that a higher rank crepant resolution conjecture can be proven using the same arguments. To do so, one should either define higher rank Bryan–Steinberg invariants directly or declare them to be whatever the wall-crossing yields.

Rationality of higher rank PT invariants

Applying similar arguments to the category $\mathtt{A}_\mu$ as those used in the proof of Theorem 1.4.11 should yield the rationality of the generating series of higher rank stable pair invariants. However, just as in the case of varieties treated in [Tod16a], the derived dual is *not* expected to induce a symmetry of this rational function.

Beyond hard Lefschetz

A natural question would be if similar techniques allow one to prove a comparison result for for Calabi–Yau orbifolds that do not satisfy the hard Lefschetz condition. An immediate problem that occurs is the fact that the McKay equivalence Φ no longer induces an equivalence between $\mathrm{Coh}(\mathcal{X})$ and a *single* tilt of $\mathrm{Coh}(Y)$; see for example [CCL17]. It does still, however, send $\Phi(\mathcal{O}_Y) = \mathcal{O}_{\mathcal{X}}$.

However, work of M. Brown and I. Shipman [BS17] shows that these categories are in fact linked by a double tilt in the local setting of $[\mathbf{C}^3/G]$ where is a finite subgroup. In general, however, it is not known that any two hearts of bounded t-structures on the bounded derived category of a smooth projective Calabi–Yau threefold (or orbifold) are related by a sequence of tilts.

Example 1.5.1. A first example of such a geometry violating the hard Lefschetz condition is the quotient singularity of type $\frac{1}{3}(1,1,1)$ at the origin of $\mathbf{C}^3$. The corresponding distinguished crepant resolution is

$$\mathbf{Z}_3\text{-Hilb}(\mathbf{C}^3) \cong \mathrm{Tot}(\omega_{\mathbf{P}^2}) \to \mathbf{P}^2,$$

the total space of the canonical bundle of the exceptional locus $E = \mathbf{P}^2$. In [Tod16b], Y. Toda has worked out the closely related PT invariants [PT09] in this situation, which naturally involves surface classes supported on the exceptional locus $\mathbf{P}^2$. He crucially uses results of A. Bayer and E. Macrì [BM11] describing the Bridgeland stability manifold of local $\mathbf{P}^2$. Even a generalisation of their results to local $\mathbf{P}^1 \times \mathbf{P}^1$ seems challenging, since the corresponding classification of Chern characters of stable objects is not yet known.

Non-multi-regular Bryan–Steinberg

The proof of the crepant resolution conjecture holds for any curve class $\beta \in \mathrm{N}_1(\mathcal{X})$, multi-regular or not. The multi-regularity property is only used in the final identification with Bryan–Steinberg invariants. For a general curve class, we thus obtain a *definition* of Bryan–Steinberg invariants in the non-multi-regular case. It would be interesting to see how they look like, and in what way they implicate the exceptional divisor in the curve counts.

The other DT/PT correspondence

Using J. Calabrese's formula (1.2.24), one can compute the generating series $\mathrm{DT}_0(\mathcal{X})$ explicitly for the various singularity types of hard Lefschetz orbifolds.

1.5.2 Motivic curve counting

Since the Hall algebra is motivic, all proofs directly yield the corresponding motivic statements once the integration map is lifted to the motivic level. Examples are the flop formula, the (higher rank) DT/PT correspondence, and the motivic crepant resolution conjecture.

1.5.3 Categorification and BPS invariants

The goal is to find objects, say graded vector spaces obtained from a cohomology theory, whose (Euler) invariants yield the DT, PT, or GW numbers. In the threefold case, for a fixed curve class, all are conjectured to be determined by finitely many *BPS numbers*, see for example [PT10]. Davison–Meinhardt two-step categorify BPS numbers to certain perverse sheaves, such that the alternating sum of the dimensions of their cohomologies equal the 'classical' BPS numbers. It would be interesting to understand how our work interacts with theirs.

1.6 Conventions

Throughout this thesis, we work over the field $\mathbf{C}$ of complex numbers. We adhere to the following conventions:

1. All rings, varieties, schemes, and stacks will be locally of finite type over $\mathbf{C}$,
2. A variety is a reduced separated scheme of finite type
3. All abelian and triangulated categories are $\mathbf{C}$-linear,
4. All functors are $\mathbf{C}$-linear,
5. We write $\mathrm{D}(\mathrm{Y}) = \mathrm{D}^b \operatorname{Coh}(\mathrm{Y})$ for the bounded derived category of coherent sheaves on a scheme or Deligne–Mumford stack Y,
6. We write $\mathrm{D}^{[a,b]}(\mathrm{Y}) \subset \mathrm{D}(\mathrm{Y})$ for the full triangulated subcategory of $\mathrm{D}(\mathrm{Y})$ such that any $\mathrm{E} \in \mathrm{D}^{[a,b]}(\mathrm{Y})$ has vanishing cohomology outside of $[a, b]$, i.e., $\mathrm{H}^i(\mathrm{E}) = 0$ for $i \notin [a, b]$.
7. We write $\operatorname{Coh}_i(\mathrm{M})$ (resp. $\operatorname{Coh}_{\leq i}(\mathrm{M})$) for the full subcategory of coherent sheaves on M of pure dimension i (resp. of dimension at most i).
8. By 'stack' we mean algebraic stack in the sense of Artin,
9. Finally, if N is an abelian group we write $\mathrm{N}_{\mathbf{Q}}, \mathrm{N}_{\mathbf{R}}$, and $\mathrm{N}_{\mathbf{C}}$ in place of the more correct $\mathrm{N} \otimes_{\mathrm{Z}} \mathbf{Q}, \mathrm{N} \otimes_{\mathrm{Z}} \mathbf{R}$, and $\mathrm{N} \otimes_{\mathrm{Z}} \mathbf{C}$ respectively.

Chapter 2

Preliminaries

2.1 Homological algebra and stability

We collect some notions from homological algebra and the theory of stability conditions that underpin much of the machinery that is to follow. Throughout, Y denotes a Gorenstein scheme or an orbifold, and $\mathtt{A}$ denotes an abelian category. By orbifold we mean a smooth Deligne–Mumford stack with generically trivial stabilisers.

2.1.1 The Grothendieck group

The numerical Grothendieck group of an abelian category has many uses: it is essential in bookkeeping of counting invariants of objects in $\mathtt{A}$, it is a natural choice of domain for a stability function on $\mathtt{A}$, and it provides a grading for the motivic Hall algebra of $\mathtt{A}$.

Definition 2.1.1. The *Grothendieck group* of an abelian category $\mathtt{A}$ is the abelian group

$$\mathrm{K}(\mathtt{A}) := \bigoplus_{\mathrm{M}\in\mathtt{A}} \mathbf{Z}\cdot[\mathrm{M}]/\sim \tag{2.1.1}$$

generated by isomorphism classes of objects in $\mathtt{A}$, modulo the relation $[\mathrm{B}] = [\mathrm{A}] + [\mathrm{C}]$ if the objects $\mathrm{A}, \mathrm{B}, \mathrm{C} \in \mathtt{A}$ fit into a short exact sequence $0 \to \mathrm{A} \to \mathrm{B} \to \mathrm{C} \to 0$.

Remark 2.1.2. The Grothendieck group $\mathrm{K}(\mathtt{T})$ of a triangulated category $\mathtt{T}$ is defined analogously by instead imposing a relation for each exact triangle.

Lemma 2.1.3. Let $\mathtt{A}$ be the heart of a bounded t-structure on a triangulated category $\mathtt{T}$. The canonical homomorphism $\phi\colon \mathrm{K}(\mathtt{A}) \to \mathrm{K}(\mathtt{T})$ is an isomorphism of abelian groups.

Proof. Note that ϕ is induced by the inclusion $\mathtt{A} \hookrightarrow \mathtt{T}$. Since the t-structure is bounded, we have $\mathrm{H}^i(\mathrm{E}) = 0$ for all $\mathrm{E} \in \mathtt{T}$ and all $|i| \gg 0$. As a consequence, the long exact cohomology sequence implies that the homomorphism

$$\psi\colon \mathrm{K}(\mathtt{T}) \to \mathrm{K}(\mathtt{A}), \quad \mathrm{E} \mapsto \sum_{i\in\mathbf{Z}} (-1)^i [\mathrm{H}^i(\mathrm{E})] \tag{2.1.2}$$

is well-defined. Since ϕ and ψ are inverses, the result follows. □

Definition 2.1.4. A triangulated category $\mathtt{T}$ is called *proper* if for all objects $\mathrm{E}, \mathrm{F} \in \mathtt{T}$

$$\sum_{i\in\mathbf{Z}} \dim \mathrm{Hom}(\mathrm{E}, \mathrm{F}[i]) < \infty. \tag{2.1.3}$$

Example 2.1.5. The triangulated category of interest is $\mathrm{D}(\mathrm{Y}) = \mathrm{D}^b(\mathrm{Coh}(\mathrm{Y}))$, the bounded coherent derived category. It contains two important subcategories. The first is $\mathrm{D_c}(\mathrm{Y})$, consisting of complexes with compact (i.e. proper) support. The second is $\mathrm{Perf}(\mathrm{Y})$, the subcategory of *perfect* complexes, which by definition are those quasi-isomorphic to bounded complexes of locally free sheaves.

When Y is proper, $\mathrm{D_c}(\mathrm{Y}) = \mathrm{D}(\mathrm{Y})$. When Y is smooth, and hence satisfies the resolution property[1] [Tot04, Thm. 1.2], $\mathrm{Perf}(\mathrm{Y}) = \mathrm{D}(\mathrm{Y})$. Thus $\mathrm{Perf}(\mathrm{Y})$ is an example of a proper triangulated category when Y is proper, and $\mathrm{D}(\mathrm{Y})$ is an example when Y is additionally smooth.

Finally, recall that the (derived) pullback of a perfect complex is always perfect and that *Serre duality* is given by the auto-equivalence $\mathrm{S_Y}(-) = (- \otimes \omega_{\mathrm{Y}})[\dim \mathrm{Y}]$.

Definition 2.1.6. The *Euler pairing* of $\mathrm{P} \in \mathrm{Perf}(\mathrm{Y})$ and $\mathrm{E} \in \mathrm{D}_c(\mathrm{Y})$ is defined as

$$\chi(\mathrm{P}, \mathrm{E}) \coloneqq \sum_i (-1)^i \dim \mathrm{Hom_Y}(\mathrm{P}, \mathrm{E}[i]) \in \mathbf{Z}.$$

Note that it is well-defined and descends to a bilinear form $\chi\colon \mathrm{K}_p(\mathrm{Y}) \otimes \mathrm{K}_c(\mathrm{Y}) \to \mathbf{Z}$ where we define the Grothendieck groups $\mathrm{K}_p(\mathrm{Y}) \coloneqq \mathrm{K}(\mathrm{Perf}(\mathrm{Y}))$ and $\mathrm{K}_c(\mathrm{Y}) \coloneqq \mathrm{K}(\mathrm{D}_c(\mathrm{Y}))$.

We identify classes of objects that have identical pairings with all perfect objects.

Definition 2.1.7. Let $\mathrm{E} \in \mathrm{D}_c(\mathrm{Y})$ be a complex.

1. We call E *numerically trivial* if $\chi(\mathrm{P}, \mathrm{E}) = 0$ for all $\mathrm{P} \in \mathrm{K}_p(\mathrm{Y})$. Since Y is Gorenstein ω_{Y} is a line bundle, so this is equivalent to $\chi(\mathrm{E}, \mathrm{P}) = 0$ by Serre duality.

2. We write $\mathrm{N}(\mathrm{Y})$ for the *numerical* Grothendieck group, which is the quotient of $\mathrm{K}_c(\mathrm{Y})$ by the subgroup generated by all numerically trivial complexes.

For $\mathrm{E} \in \mathrm{D}_c(\mathrm{Y})$, we write $[\mathrm{E}] \in \mathrm{N}(\mathrm{Y})$ for its numerical class.

[1] This means that Y has 'enough' locally free sheaves: every coherent sheaf on Y is a quotient of one.

Remark 2.1.8. If Y is a smooth and projective variety, then $\mathrm{N}(\mathrm{Y})$ has finite rank by the Hirzebruch–Riemann–Roch Theorem. Indeed, given $\mathrm{E}, \mathrm{F} \in \mathrm{D}(\mathrm{Y})$ it states that $\chi(\mathrm{E}, \mathrm{F}) = \int_{\mathrm{Y}} \mathrm{ch}(\mathrm{E})^{\vee} \mathrm{ch}(\mathrm{F}) \mathrm{Td}(\mathrm{Y})$ where ch denotes the Chern character in cohomology. In particular, we obtain a surjective morphism

$$\mathrm{H}^{*}(\mathrm{Y}, \mathbf{Q}) \supset \operatorname{im} \mathrm{ch} = \mathrm{K}(\mathrm{Y})_{\mathbf{Q}} / \ker \mathrm{ch} \twoheadrightarrow \mathrm{N}(\mathrm{Y})_{\mathbf{Q}}. \tag{2.1.4}$$

Since $\mathrm{H}^{*}(\mathrm{Y}, \mathbf{Q})$ is a finite-dimensional $\mathbf{Q}$-vector space, the claim follows.

Using B. Toën's Hirzebruch–Riemann–Roch Theorem [Toe99], the same reasoning shows that $\mathrm{N}(\mathrm{Y})$ also has finite rank when Y is an orbifold. The precise statement of HRR in the context of orbifolds can for example be found in [BCY12, Thm. 35].

Now we are in a position to define a rank function in the generality we need.

Definition 2.1.9. Assume Y is irreducible with generically trivial stabiliser groups. Given a complex $\mathrm{F} \in \mathrm{D}_c(\mathrm{Y})$, its *rank* is $\mathrm{rk}(\mathrm{F}) \coloneqq \chi(\mathrm{F}, \mathcal{O}_p) \in \mathbf{Z}$ where $p \in \mathrm{Y}$ is any non-stacky point. This notion descends to an additive map $\mathrm{rk}\colon \mathrm{N}(\mathrm{Y}) \to \mathbf{Z}$, since any such $\mathcal{O}_p$ has a locally free resolution given by the Koszul whence $[\mathcal{O}_p] \in \mathrm{K}_p(\mathrm{Y})$.

The group $\mathrm{N}(\mathrm{Y})$ has a natural filtration by the dimension of support. We write $\mathrm{Coh}_{\leq d}(\mathrm{Y}) \subset \mathrm{Coh}(\mathrm{Y})$ for the subcategory of sheaves supported in dimension at most d. We then define $\mathrm{N}_{\leq d}(\mathrm{Y}) < \mathrm{N}(\mathrm{Y})$ for the subgroup generated by classes of (compactly supported) sheaves $\mathrm{F} \in \mathrm{Coh}_{\leq d}(\mathrm{Y})$. We write

$$\mathrm{N}_d(\mathrm{Y}) \coloneqq \mathrm{N}_{\leq d}(\mathrm{Y}) / \mathrm{N}_{\leq d-1}(\mathrm{Y}) \tag{2.1.5}$$

for the associated graded piece of dimension d.

Example 2.1.10. Note that $\mathrm{N}_{\leq 0}(\mathrm{Y}) = \mathrm{N}_0(\mathrm{Y})$. Moreover, if Y is a smooth and projective variety, we have an isomorphism $\chi(\mathcal{O}_{\mathrm{Y}}, -)\colon \mathrm{N}_0(\mathrm{Y}) \to \mathbf{Z}$ showing that $\mathrm{N}_0(\mathrm{Y})$ is generated by the class of a point. However, $\mathrm{N}_0(\mathrm{Y})$ will be larger when Y is a DM stack, since skyscraper sheaves supported at stacky points with different equivariant structures will in general have different numerical class.

Clearly, the subgroups $\mathrm{N}_{\leq d}(\mathrm{Y}) \subset \mathrm{N}(\mathrm{Y})$ are free abelian groups of finite rank, for all d. Similarly, one can show that the quotients $\mathrm{N}_d(\mathrm{Y})$ are also free.

Remark 2.1.11. The association of the numerical Grothendieck group is functorial for proper morphisms. Let $f\colon \mathrm{Y} \to \mathrm{Y}'$ be a proper morphism, then derived pushforward induces a homomorphism between Grothendieck groups $\mathbf{R}f_*\colon \mathrm{K}(\mathrm{Y}) \to \mathrm{K}(\mathrm{Y}')$. This map descends to the numerical Grothendieck group, i.e.,

$$\mathbf{R}f_*\colon \mathrm{N}(\mathrm{Y}) \to \mathrm{N}(\mathrm{Y}') \tag{2.1.6}$$

as it preserves numerically trivial classes. Indeed, suppose E is a numerically trivial complex on Y. Then for any $P \in \mathrm{Perf}(Y')$, $\mathbf{L}f^*P$ is also perfect. From the adjunction

$$\mathbf{R}\mathrm{Hom}_{Y'}(P, \mathbf{R}f_*E) = \mathbf{R}\mathrm{Hom}_Y(\mathbf{L}f^*P, E)$$

we deduce that $\chi(P, \mathbf{R}f_*E) = \chi(\mathbf{L}f^*P, E) = 0$. The claim now follows. Moreover, note that $\mathbf{R}f_*(N_{\leq d}(Y)) \subset N_{\leq d}(Y')$ as f does not increase the dimension of the support.

Remark 2.1.12. Finally, if Y is a smooth and projective variety, the inclusions

$$i_d \colon N_0(Y) \hookrightarrow N_{\leq d}(Y) \tag{2.1.7}$$

are naturally split by the Euler characteristic $\chi(\mathcal{O}_Y, -) \colon N_{\leq d}(Y) \twoheadrightarrow N_0(Y)$, for every $d \geq 1$. If $d = 1$, this yields a canonical integral splitting

$$s \colon N_{\leq 1}(Y) \xrightarrow{\sim} N_0(Y) \oplus N_1(Y) \tag{2.1.8}$$

that decomposes any class $\gamma \in N_{\leq 1}(Y)$ as $s(\gamma) = (\gamma_0, \gamma_1)$ such that $\chi(\mathcal{O}_Y, \gamma_1) = 0$. It turns out that finding such an integral splitting is not always possible when Y is a smooth Deligne–Mumford stack with projective coarse moduli space; see Remark 2.1.34.

2.1.2 Torsion pairs and tilting

To prove our universal wall-crossing formula, we use the construction of tilting at a torsion pair of [HRS96]. We recall it here.

Definition 2.1.13. If $\mathtt{B}$ is an abelian category, a *torsion pair* consists of a pair of additive subcategories $(\mathtt{T}, \mathtt{F})$ such that

1. $\mathrm{Hom}(T, F) = 0$, for all $T \in \mathtt{T}$, $F \in \mathtt{F}$;
2. every object $E \in \mathtt{B}$ fits into a short exact sequence

$$0 \to T_E \to E \to F_E \to 0$$

with $T_E \in \mathtt{T}$ and $F_E \in \mathtt{F}$.

We write $\mathtt{B} = \langle T, F \rangle$ and call $\mathtt{T}$ the *torsion subcategory* and $\mathtt{F}$ the *torsion free subcategory.* Note that the first condition implies that the sequence in the second condition is unique.

The associations $E \mapsto T_E$ and $E \mapsto F_E$ extend to functors that are right and left adjoint to the inclusion functors of $\mathtt{T}$ and $\mathtt{F}$ in $\mathtt{B}$ respectively.

Example 2.1.14. The original example of a torsion pair is the following. Let C be a smooth projective curve. Any coherent sheaf E on C has a unique maximal torsion coherent subsheaf

$$0 \to \mathrm{T}_0(\mathrm{E}) \to \mathrm{E} \to \mathrm{E}/\mathrm{T}_0(\mathrm{E}) \to 0.$$

The quotient is locally free since the local rings of C are PIDs. There are no maps from torsion sheaves to locally free sheaves, so this defines a torsion pair inside Coh(C).

It is easy to construct torsion pairs on *noetherian* abelian categories.

Definition 2.1.15. An abelian category $\mathtt{B}$ is *noetherian* if for any object $\mathrm{E} \in \mathtt{B}$ any ascending chain of subobjects $\mathrm{E}_0 \subset \mathrm{E}_1 \subset \ldots \subset \mathrm{E}_n \subset \ldots \subset \mathrm{E}$ stabilizes.

Lemma 2.1.16. The abelian category of coherent sheaves on a noetherian scheme or on a noetherian Deligne–Mumford stack is noetherian.

Proof. Suppose to the contrary that there is a coherent sheaf F on a noetherian scheme Y that contains an infinite ascending chain of coherent subsheaves

$$\mathrm{E}_0 \subset \mathrm{E}_1 \subset \ldots \subset \mathrm{E}_n \subset \ldots \subset \mathrm{F}. \tag{2.1.9}$$

Any affine open of Y is the spectrum of a noetherian ring, and we need finitely many to cover Y. Pulled back to an affine, we find an ascending sequence of finitely generated submodules

$$\mathrm{M}_0 \subset \mathrm{M}_1 \subset \ldots \subset \mathrm{M}_n \subset \ldots \subset \mathrm{N}$$

of a noetherian module N. By the ascending chain condition of N, this pulled back chain stabilizes. Repeating this argument for each of the members of a finite affine cover of Y, we deduce that the original chain in equation (2.1.9) stabilizes as well.

If Y is a noetherian Deligne–Mumford stack, then the claim follows by pulling back such a chain to a noetherian atlas $a \colon \mathrm{A} \twoheadrightarrow \mathrm{Y}$, where a is an étale surjective morphism. ☐

To construct torsion pairs, there is the following result.

Lemma 2.1.17. Let $\mathtt{B}$ be a noetherian abelian category and let $\mathtt{T}$ be a full subcategory that is closed under extensions and quotients. Let

$$\mathtt{F} := \mathtt{T}^{\perp} = \{\mathrm{F} \in \mathtt{B} \mid \mathrm{Hom}(\mathrm{T}, \mathrm{F}) = 0 \text{ for all } \mathrm{T} \in \mathtt{T}\}.$$

Then $(\mathtt{T}, \mathtt{F})$ is a torsion pair on $\mathtt{B}$.

Proof. See e.g.[Tod13, Lem. 2.15]. ☐

The abelian category $\mathtt{B}$ defines the standard t-structure on $D(\mathtt{B})$. In the presence of a torsion pair, one may construct a different t-structure via the process of *tilting*.

Proposition 2.1.18. Let $(\mathtt{T},\mathtt{F})$ be a torsion pair on the abelian category $\mathtt{B}$, and let $H^i(E)$ denote the ith cohomology object of $E \in D(\mathtt{B})$ with respect to the standard t-structure on $D(\mathtt{B})$. Then

$$\begin{aligned} \mathtt{B}^\flat &:= \langle \mathtt{F}[1], \mathtt{T}\rangle \\ &= \{E \in D(\mathtt{B}) \mid H^{-1}(E) \in \mathtt{F},\, H^0(E) \in \mathtt{T},\, H^i(E) = 0 \text{ if } i \neq -1, 0\} \end{aligned} \tag{2.1.10}$$

defines a heart of a bounded t-structure on $D(\mathtt{B})$. In particular, $\mathtt{B}^\flat$ is abelian.

Proof. This is [HRS96, Cor. 2.2.(a)]. □

Remark 2.1.19. A short exact sequence $0 \to F \to G \to E \to 0$ in $\mathtt{B}^\flat$ is nothing but an exact triangle $F \to G \to E \to F[1]$ in $D(\mathtt{B})$, with $E, F, G \in \mathtt{B}^\flat$. Put differently,

$$\mathrm{Ext}^1_{\mathtt{B}^\flat}(E,F) = \mathrm{Hom}_{D(\mathtt{B})}(E, F[1]). \tag{2.1.11}$$

Similarly, $\mathrm{Hom}_{\mathtt{B}^\flat}(E,F) = \mathrm{Hom}_{D(\mathtt{B})}(E,F)$ since $\mathtt{B}^\flat$ is a full subcategory. Note, however, that higher extensions may differ; see [Bri05, Ex. 3.7] for a geometric example.

2.1.3 Slope stability on orbifolds

The theory of stability conditions is by now a rich and important subject, with applications in moduli theory, birational geometry, and the study of derived categories. In this thesis, we will only make use of a very small part of this machinery, even though the underlying philosophy of wall-crossing induced by a change of stability condition, embodied by formulas in the motivic Hall algebra, is fundamental to all our results.

To this extent we record here our bare-bones notion of a stability condition, and the two examples of it we will use: an extension of slope stability to orbifolds due to F. Nironi [Nir08], and a version of weak stability condition in the sense of D. Joyce [Joy07]. Both of these stability conditions are defined on the abelian category $\mathrm{Coh}_{\leq 1}(\mathcal{X})$, where $\mathcal{X}$ is a (smooth) orbifold in the sense of Definition 1.2.13.

For the reader's convenience, we recall some well-known properties of orbifolds, i.e., smooth Deligne–Mumford stacks with generically trivial stabilisers by Definition 1.2.13.

Proposition 2.1.20. Let $\mathcal{X}$ be an orbifold. Then

1. $\mathcal{X}$ has finite stabilisers, i.e., the natural morphism $I_{\mathcal{X}} \to \mathcal{X}$ is a finite morphism, where $I_{\mathcal{X}} \cong \mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X}$ is the inertia stack,

2. $\mathcal{X}$ has a coarse moduli space $\pi\colon \mathcal{X} \to \mathrm{X}$ and π is a proper and quasi-finite morphism,

3. the natural morphism $\mathcal{O}_{\mathrm{X}} \to \pi_*(\mathcal{O}_{\mathcal{X}})$ is an isomorphism,

4. the functor $\pi_*\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathrm{X})$ is exact and maps injective sheaves to flasque sheaves, hence $\mathrm{H}^i(\mathcal{X}, \mathrm{F}) \cong \mathrm{H}^i(\mathrm{X}, \pi_*\mathrm{F})$ for every $\mathrm{F} \in \mathrm{QCoh}(\mathcal{X})$; in particular, the Euler characteristic may be computed on the coarse space $\chi(\mathcal{O}_{\mathcal{X}}, \mathrm{F}) = \chi(\mathcal{O}_{\mathrm{X}}, \pi_*\mathrm{F})$,

5. the functor π_* restricts to an exact functor $\mathrm{Coh}(\mathcal{X}) \to \mathrm{Coh}(\mathrm{X})$,

6. if F is a locally free sheaf on $\mathcal{X}$, then $\pi_*\mathrm{F}$ is locally free on X.

Recall that X being the coarse moduli space of $\mathcal{X}$ means that π is a bijection on geometric points and that if $g\colon \mathcal{X} \to \mathrm{Z}$ is another morphism with this property, where Z is a scheme, then there exists a unique morphism $g'\colon \mathrm{X} \to \mathrm{Z}$ such that $g' \circ \pi = g$.

Proof. See the first section of [Nir08] and references therein. □

We fix once and for all what is meant by a *(quasi-)projective* orbifold in accordance with [Kre09, Thm. 5.3]. Recall that our orbifolds are always assumed to be smooth.

Definition 2.1.21. A *(quasi-)projective* orbifold is a smooth proper Deligne–Mumford stack with generically trivial stabilisers such that its coarse moduli space $\pi\colon \mathcal{X} \to \mathrm{X}$ is (quasi-)projective.

Finally, we record the following philosophically satisfying statement.

Proposition 2.1.22. Let X be a finite type (quasi-)projective scheme with at worst quotient singularities. Then there exists a (quasi-)projective orbifold $\mathcal{X}$ with coarse moduli space $g\colon \mathcal{X} \to \mathrm{X}$.

Proof. This follows for example from [Vis89, Prop. 2.8]. □

Let $\mathcal{X}$ be an orbifold. We consider stability conditions on the category $\mathrm{Coh}_{\le 1}(\mathcal{X})$.

Definition 2.1.23. A *stability condition* on $\mathrm{Coh}_{\le 1}(\mathcal{X})$ consists of a *slope function* $\mu\colon \mathrm{N}_{\le 1}(\mathcal{X}) \to \mathrm{S}$ where $(\mathrm{S}, \le)$ is a totally ordered set, such that

1. the slope μ satisfies the *see-saw property*, i.e., given a short exact sequence $0 \to \mathrm{A} \to \mathrm{B} \to \mathrm{C} \to 0$ in $\mathrm{Coh}_{\le 1}(\mathcal{X})$ we have either

$$\mu(\mathrm{A}) < \mu(\mathrm{B}) < \mu(\mathrm{C}) \quad \text{or} \quad \mu(\mathrm{A}) = \mu(\mathrm{B}) = \mu(\mathrm{C}) \quad \text{or} \quad \mu(\mathrm{A}) > \mu(\mathrm{B}) > \mu(\mathrm{C});$$

2. the category $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ has the *Harder–Narasimhan property* with respect to μ, i.e., any sheaf $\mathrm{F} \in \mathrm{Coh}_{\leq 1}(\mathcal{X})$ admits a filtration in $\mathrm{Coh}_{\leq 1}(\mathcal{X})$,

$$0 = \mathrm{F}_0 \subset \mathrm{F}_1 \subset \ldots \subset \mathrm{F}_{n-1} \subset \mathrm{F}_n = \mathrm{F},$$

   such that each quotient factor $\mathrm{Q}_i = \mathrm{F}_i/\mathrm{F}_{i-1}$ is semistable of descending slope $\mu(\mathrm{Q}_1) > \mu(\mathrm{Q}_2) > \ldots > \mu(\mathrm{Q}_n)$.

A sheaf $\mathrm{F} \in \mathrm{Coh}_{\leq 1}(\mathcal{X})$ is *stable* if for all non-trivial proper subsheaves $0 \neq \mathrm{E} \subset \mathrm{F}$

$$\mu(\mathrm{E}) < \mu(\mathrm{F})$$

or, equivalently, $\mu(\mathrm{F}) < \mu(\mathrm{F}/\mathrm{E})$ or $\mu(\mathrm{E}) < \mu(\mathrm{F}/\mathrm{E})$ by the see-saw property. To obtain the notion of *semistability*, replace each strict inequality $<$ by a weak one $\leq$.

Remark 2.1.24. Let μ be a slope function on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$. Since $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ is noetherian in the sense of 2.1.15, one can show that Harder–Narasimhan filtrations with respect to μ exist when $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ is μ-artinian, i.e., when any chain of subobjects

$$\mathrm{F}_0 \supset \mathrm{F}_1 \supset \mathrm{F}_2 \supset \ldots \tag{2.1.12}$$

in $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ such that $\mu(\mathrm{F}_{i-1}) \leq \mu(\mathrm{F}_i)$ stabilizes; see e.g. [Joy07, Thm. 4.4].

As an example, we describe F. Nironi's extension of slope stability to tame Deligne–Mumford stacks with projective coarse moduli space as in [Nir08]. In fact, he introduces the analogue of Gieseker stability, but the two are equivalent on $\mathrm{Coh}_{\leq 1}$.

Example 2.1.25. Let Y be a smooth projective variety. We recall the construction of Gieseker or slope stability on $\mathrm{Coh}_{\leq 1}(\mathrm{Y})$, with respect to a fixed very ample line bundle $\mathcal{O}_{\mathrm{Y}}(1)$ of class H, following [HL10, §1.6].

The Hilbert polynomial of $\mathrm{F} \in \mathrm{Coh}_{\leq 1}(\mathrm{Y})$ is the linear polynomial given by

$$p_{\mathrm{F}}(k) := \chi(\mathrm{Y}, \mathrm{F}(k)) = a_1(\mathrm{F})k + a_0(\mathrm{F}) \tag{2.1.13}$$

where $a_1(\mathrm{F}), a_0(\mathrm{F}) \in \mathbf{Z}$ and $k \in \mathbf{Z}$; the polynomiality follows for example from [HL10, Lem. 1.2.1] and the linearity follows since $\dim(\mathrm{F}) = 1$. The *slope* of the sheaf F is

$$\mu_{\mathrm{H}}(\mathrm{F}) := \frac{a_0(\mathrm{F})}{a_1(\mathrm{F})} \in \mathbf{Q} \cup \{\infty\} \tag{2.1.14}$$

where $\mu_{\mathrm{H}}(\mathrm{F}) = \infty$ only if $a_1(\mathrm{F}) = 0$. The latter occurs if and only if $\mathrm{F} \in \mathrm{Coh}_0(\mathrm{Y})$. It is easy to see that μ_{H} satisfies the see-saw property, and we say that F is *slope (semi)stable* if the usual condition of Definition 2.1.23 holds.

Note that we only consider orbifolds in the sense of Definition 1.2.13.

Remark 2.1.26. On an orbifold $\mathcal{X}$, the above definition of stability needs to be modified. The problem is that a non-zero sheaf F, supported on the stacky locus of $\mathcal{X}$, that has a non-trivial equivariant structure can have vanishing Hilbert polynomial $p_{\mathrm{F}}(k) = 0$. This is due to the property $\chi(\mathcal{X}, \mathrm{F} \otimes \pi^* \mathcal{O}_{\mathrm{X}}(k)) = \chi(\mathrm{X}, \pi_* \mathrm{F} \otimes \mathcal{O}_{\mathrm{X}}(k))$ that follows from Proposition 2.1.20.4.

To illustrate the issue, consider the orbifold $\mathcal{X} = [\mathrm{T}/\mathbf{Z}_2]$ where T is the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ on $\mathbf{P}^1$ and $\mathbf{Z}_2$ acts fibre-wise via $(x, y) \mapsto (-x, -y)$. Its stacky locus is the zero section $\mathrm{C} \subset \mathcal{X}$. Let $\{\rho^+, \rho^-\}$ denote the trivial and non-trivial irreducible representations of $\mathbf{Z}_2$, and consider $\mathcal{O}_{\mathrm{C}}^- = \mathcal{O}_{\mathrm{C}} \otimes \rho^-$ the structure sheaf of C with the non-trivial equivariant structure. We will compute its Hilbert polynomial.

It suffices to consider the induced morphism $\pi\colon [\mathbf{P}^1/\mathbf{Z}_2] \to \mathbf{P}^1/\mathbf{Z}_2 \subset \mathrm{X}$. A sheaf on $[\mathbf{P}^1/\mathbf{Z}_2]$ is equivalent to a sheaf on $\mathbf{P}^1$ with a $\mathbf{Z}_2$-equivariant structure. Pushing forward over π takes the $\mathbf{Z}_2$-invariant part. It follows that $\pi_*(\mathcal{O}_{\mathrm{C}}^-(k)) = 0$ whence $\chi(\mathcal{X}, \mathcal{O}_{\mathrm{C}}^-(k)) = 0$ by Proposition 2.1.20.4.

Remark 2.1.27. Roughly speaking, the usual Hilbert polynomial of a sheaf supported on the stacky locus only picks up the part of the sheaf with the trivial equivariant structure. More precisely, in the above example $\mathcal{X} = [\mathbf{P}^1/\mathbf{Z}_2]$ is the trivial gerbe over $\mathbf{P}^1$ banded by $\mathbf{Z}_2$. Any sheaf F on this gerbe has an eigensheaf decomposition

$$\mathrm{F} = \mathrm{F}_+ \otimes \rho^+ \oplus \mathrm{F}_- \otimes \rho^- \tag{2.1.15}$$

according to the irreducible representations of the banding group $\mathbf{Z}_2$. Pushing F forward over π kills all but the trivial eigensheaf. Let $p \in \mathcal{X} = [\mathbf{P}^1/\mathbf{Z}_2]$. Note, however, that

$$\chi(\mathcal{O}_{\mathcal{X}} \otimes \rho^-, \mathcal{O}_p^-) = \chi(\mathcal{O}_{\mathcal{X}} \otimes \rho^+, \mathcal{O}_p^+). \tag{2.1.16}$$

This suggests that replacing $\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}} \otimes \rho^+$ with $\mathrm{V} = \mathcal{O}_{\mathcal{X}} \otimes \mathbf{C}\mathbf{Z}_2$ corrects the vanishing. In this case, V is an example of a *generating vector bundle* in the sense of M. Olsson and J. Starr [OS03], which roughly means that V is π-very ample.

We describe their constructing of a generating sheaf for orbifolds in our sense.

Definition 2.1.28. Let V be a locally free sheaf on $\mathcal{X}$, and define the functor

$$\mathrm{G}_{\mathrm{V}}\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathrm{X}), \quad \mathrm{F} \mapsto \mathrm{G}_{\mathrm{V}}(\mathrm{F}) \coloneqq \pi_* \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{X}}}(\mathrm{V}, \mathrm{F}). \tag{2.1.17}$$

Note that $\mathrm{G}_{\mathrm{V}}(-) \cong \pi_*(\mathrm{V}^\vee \otimes -)$ is an exact functor. Consider the double left adjoint of the identity morphism $\mathbf{1}\colon \pi_*(V^\vee \otimes F) \to \pi_*(V^\vee \otimes F)$, which we denote by

$$\theta_V(F)\colon \pi^*\pi_* \underline{\mathrm{Hom}}(V,F) \otimes V \to F. \tag{2.1.18}$$

1. V is a *generator for* $F \in \mathrm{QCoh}(\mathcal{X})$ if $\theta_V(F)$ is surjective,
2. V is a *generating sheaf* if it is a generator for every $F \in \mathrm{QCoh}(\mathcal{X})$.

Example 2.1.29. The following is the typical example of a generating sheaf, generalising the above example for $\mathbf{Z}_2$. Let G be a finite group, and let $\mathcal{X} = BG \times_{\mathrm{Spec}\,\mathbf{Z}} \mathrm{Spec}\,\mathbf{C}$ be its classifying stack. It is a smooth DM stack of finite type.

Recall the equivalence of abelian categories

$$\phi\colon \mathrm{Coh}(\mathcal{X}) \xrightarrow{\sim} \mathrm{Mod}(\mathbf{C}[G]), \tag{2.1.19}$$

where $\mathrm{Mod}(\mathbf{C}[G])$ is the category of finite-dimensional representations of the group algebra $\mathbf{C}[G]$. Let V be the locally free sheaf corresponding to the regular representation of $\mathbf{C}[G]$. Then V is a generating vector bundle. The key property is the fact that

$$\mathrm{Hom}(\mathbf{C}[G], \rho_i) \neq 0 \tag{2.1.20}$$

for all irreducible representations ρ_i of G, avoiding unwanted vanishing of the Hilbert polynomial of sheaves with non-trivial equivariant structure as in Remark 2.1.26.

The following result guarantees the existence of a generating sheaf on orbifolds.

Lemma 2.1.30. Let $\mathcal{X}$ be an orbifold with a quasi-projective coarse moduli space. Then there exists a generating sheaf V on $\mathcal{X}$ that is locally free and self-dual $V \cong V^\vee$.

Proof. Since $\mathcal{X}$ satisfies the resolution property, $\mathcal{X}$ is isomorphic to a global quotient stack by [Kre09, Prop. 5.1]. As explained in [Kre09, Thm. 5.3], it follows from [OS03, §5] that this is equivalent to $\mathcal{X}$ having a locally free generating sheaf V.

Finally, it is clear from equation (2.1.18) that $V \oplus V^\vee$ is a generating sheaf given that V is one. Thus, replacing V with $V \oplus V^\vee$ if necessary, we have $V \cong V^\vee$. □

In [OS03, §5], M. Olsson and J. Starr show that vanishing of the Hilbert polynomial is the only pathology occurring in defining slope stability on orbifolds. We now show how the existence of a generating sheaf allows one to get around this pathology.

Fix a self-dual generating vector bundle V on the orbifold $\mathcal{X}$.

Definition 2.1.31. Let $\mathcal{X}$ be an orbifold with projective coarse moduli space $\pi\colon \mathcal{X} \to X$, and fix an ample line bundle A on X.[2] For $F \in \mathrm{Coh}_{\le 1}(\mathcal{X})$, the *modified Hilbert polynomial*

[2]When no confusion arises, we abuse notation and write A also for π^*A on $\mathcal{X}$.

$p_{\mathrm{F}}(k)$ is defined as

$$p_{\mathrm{F}}(k) := \chi(\mathcal{X}, \mathrm{V}^\vee \otimes \mathrm{F}(k)) = a_1(\mathrm{F})k + a_0(\mathrm{F}) \tag{2.1.21}$$

for $k \in \mathbf{Z}$, where $\mathrm{F}(k) := \mathrm{F} \otimes \mathrm{A}^{\otimes k}$. It is a linear polynomial by (4) of Proposition 2.1.20.

As χ is additive on exact sequences, the definition of p_{F} descends to the Grothendieck group of $\mathrm{Coh}_{\leq 1}(\mathcal{X})$. Moreover, $p_{\mathrm{E}}(k)$ will be identically zero for any numerically trivial complex E since $\mathrm{V} \otimes \mathrm{A}^{\otimes k}$ is locally free. Hence, $p_\gamma(k)$ is well defined on $\gamma \in \mathrm{N}_{\leq 1}(\mathcal{X})$.

The following result is [Nir08, Thm. 4.20], based on [OS03, Thm. 6.1]. It is the fundamental boundedness result regarding flat families of quotients on $\mathcal{X}$.

Theorem 2.1.32. Let $\mathcal{X}$ be an orbifold with projective coarse space X, let A be an ample line bundle on X, and let V be a generating vector bundle on $\mathcal{X}$. Let $\underline{\mathrm{Quot}}_{\mathcal{X}}(\mathrm{F}, p)$ denote the functor of quotients of $\mathrm{F} \in \mathrm{Coh}(\mathcal{X})$ of modified Hilbert polynomial $p \in \mathbf{Z}[x]$. It is represented by a projective scheme, which we denote by $\mathrm{Quot}_{\mathcal{X}}(\mathrm{F}, p)$.

We give a precise description of the coefficients of the modified Hilbert polynomial. In fact, we will use its zeroeth coefficient as a substitute for the notion of *degree* on $\mathcal{X}$.

Definition 2.1.33. Given $\gamma \in \mathrm{N}_{\leq 1}(\mathcal{X})$, we define its *degree* to be

$$\deg(\gamma) := p_\gamma(0) = a_0(\gamma) \in \mathbf{Z} \tag{2.1.22}$$

where $p_\gamma(k) = a_1(\gamma)k + a_0(\gamma)$ is the modified Hilbert polynomial of γ.

The degree descends to a homomorphism

$$\deg\colon \mathrm{N}_{\leq 1}(\mathcal{X}) \to \mathbf{Z}, \quad \gamma \mapsto \deg(\gamma) := p_\gamma(0). \tag{2.1.23}$$

Note that the definition of degree depends on the choice of generating vector bundle V.

Remark 2.1.34. There is a natural exact sequence $0 \to \mathrm{N}_0(\mathcal{X}) \to \mathrm{N}_{\leq 1}(\mathcal{X}) \to \mathrm{N}_1(\mathcal{X}) \to 0$ of free abelian groups of finite rank; note that $\mathrm{N}_1(\mathcal{X})$ is free abelian by the argument below Example 2.1.10. Contrary to the case of varieties discussed in Remark 2.1.12, there is an obstruction to splitting this exact sequence

$$\mathrm{N}_{\leq 1}(\mathcal{X}) \cong \mathrm{N}_1(\mathcal{X}) \oplus \mathrm{N}_0(\mathcal{X}) \tag{2.1.24}$$

compatibly with the degree, i.e., such that $\deg(\alpha) = \deg(c)$ if $\alpha = (\beta, c)$ under this splitting with $c \in \mathrm{N}_0(\mathcal{X})$, i.e., $\mathrm{N}_1(\mathcal{X}) \subset \ker(\deg)$. Denote the composition of the inclusion $\mathrm{N}_0(\mathcal{X}) \hookrightarrow \mathrm{N}_{\leq 1}(\mathcal{X})$ with deg by $\deg_0$. The splitting can not be made compatibly if

$$\mathrm{im}(\deg_0) \subsetneq \mathrm{im}(\deg). \tag{2.1.25}$$

However, after passing to rational coefficients (or a finite extension), such a compatible splitting exists. Henceforth, we fix an *integral* splitting, so not necessarily compatible.

To give a precise description of the leading coefficient of the polynomial $p_\gamma(k)$ for $\gamma \in \mathrm{N}_{\leq 1}(\mathcal{X})$ we use the Kleiman-trick. First, however, we introduce the following

Definition 2.1.35. We call a class $\gamma \in \mathrm{N}(\mathcal{X})$ *effective* if $\gamma = [\mathrm{F}]$ for some sheaf $\mathrm{F} \in \mathrm{Coh}(\mathcal{X})$. We write $\gamma_1 \leq \gamma_2$ if the class $\gamma_2 - \gamma_1$ is effective in $\mathrm{N}(\mathcal{X})$. An effective class in $\mathrm{N}_{\leq 1}(\mathcal{X})$ is called an *effective curve class*.

Remark 2.1.36. Any class in $\mathrm{N}(\mathcal{X})$ is the difference of two effective classes.

Lemma 2.1.37. Let $\gamma \in \mathrm{N}_{\leq 1}(\mathcal{X})$ be a one-dimensional class. Then $a_1(\gamma) = \chi(\mathrm{V}, \gamma \cdot \mathrm{A})$.

Here $\gamma \cdot \mathrm{A}$ is defined by writing $\gamma = \beta_1 - \beta_2$ with $\beta_i \geq 0$ effective. Let $\mathrm{T} \in \mathrm{Coh}_{\leq 1}(\mathcal{X})$ represent an effective class $\beta = [\mathrm{T}]$. Then we define $\beta \cdot \mathrm{A} := [\mathrm{T}|_\mathrm{A}] = [\mathrm{T}] - [\mathrm{T}(-\mathrm{A})]$.

Proof. First assume that γ is effective, so $\gamma = [\mathrm{T}]$ for some $\mathrm{T} \in \mathrm{Coh}_{\leq 1}(\mathcal{X})$. Let $k \in \mathbf{Z}$. Tensoring the short exact sequence $0 \to \mathrm{T}(-\mathrm{A}) \to \mathrm{T} \to \mathrm{T}|_\mathrm{A} \to 0$ by the line bundle $\mathrm{A}^{\otimes k}$, and applying the additive function $\chi(\mathrm{V}, -)$ shows that

$$\chi(\mathrm{V}, \mathrm{T} \otimes \mathrm{A}^{\otimes k}) = \chi(\mathrm{V}, \mathrm{T} \otimes \mathrm{A}^{\otimes k-1}) + \chi(\mathrm{V}, \mathrm{T}|_\mathrm{A} \otimes \mathrm{A}^{\otimes k}).$$

But $[\mathrm{T}] - [\mathrm{T}(-\mathrm{A})] = [\mathrm{T}|_\mathrm{A}] \equiv \gamma \cdot \mathrm{A}$ as classes in $\mathrm{N}_0(\mathcal{X})$. Thus the above equality reads $p_\gamma(k) = p_\gamma(k-1) + \chi(\mathrm{V}, \gamma \cdot \mathrm{A})$. Since $p_\gamma(k) = a_1(\gamma)k + a_0(\gamma)$ we may conclude.

For a general class γ, the result follows by bilinearity of the Euler form. □

Let $\Lambda(\mathcal{X}) \subset \mathrm{N}_1(\mathcal{X})$ denote the (convex) cone[3] spanned by effective curve classes on $\mathcal{X}$, i.e., classes of one-dimensional quotients of $\mathcal{O}_\mathcal{X}$. We call $\Lambda(\mathcal{X})_\mathbf{R}$ the *effective cone of* $\mathcal{X}$, where $\Lambda(\mathcal{X})_\mathbf{R} = \Lambda(\mathcal{X}) \otimes_\mathbf{Z} \mathbf{R}$.

Lemma 2.1.38. Let $\mathcal{X}$ be an orbifold with projective coarse moduli space $\pi\colon \mathcal{X} \to \mathrm{X}$. For any $\beta \in \Lambda(\mathcal{X})$, there are at most finitely many $\beta' \in \Lambda(\mathcal{X})$ with $\beta' \leq \beta$.

Proof. Since X is projective, the effective cone of curves $\Lambda(\mathrm{X}) \subset \mathrm{N}_1(\mathrm{X})_\mathbf{R}$ is strictly convex by [KM98, Cor. 1.19]. As a consequence, any class $\beta \in \Lambda(\mathrm{X})$ has at most finitely many decomposition into effective curve classes on X. Similarly, since $\mathrm{Coh}_0(\mathcal{X})$ is an artinian category, any effective point class has at most finitely many decompositions into effective point classes on $\mathcal{X}$. Thus, it suffices to show that the linear function

$$t\colon \mathrm{N}_1(\mathcal{X}) \to \mathrm{N}_1(\mathrm{X}) \oplus \mathrm{N}_0(\mathcal{X}), \quad \alpha \mapsto t(\alpha) := \pi_*(\alpha) \oplus (\alpha \cdot \pi^*\mathrm{A}) \tag{2.1.26}$$

[3] By this we mean $u, v \in \Lambda(\mathcal{X})$ and $a, b \in \mathbf{Z}_{\geq 0}$ implies $au + bv \in \Lambda(\mathcal{X})$; strictly speaking, $\Lambda(\mathcal{X})$ is a commutative monoid and its extension $\Lambda(\mathcal{X})_\mathbf{R}$ is a convex cone.

is injective on effective curve classes $\Lambda_1(\mathcal{X})$ in order to deduce the claim.

We claim that t is injective. We appeal to the McKay equivalence, to be introduced in section 2.4. It is an equivalence of derived categories $\Psi\colon \mathrm{D}(\mathcal{X}) \to \mathrm{D}(\mathrm{Y})$, where $f\colon \mathrm{Y} \to \mathrm{X}$ is a resolution of singularities, that commutes with pushing forward $\pi_* = \mathbf{R}f_* \circ \Psi$.

We reason as follows. Let $\alpha \in \mathrm{N}_1(\mathcal{X})$ be a class such that $t(\alpha) = \pi_*(\alpha) \oplus \alpha \cdot \pi^*(\mathrm{A}) = 0$. On the one hand, we have $0 = \Psi(\alpha \cdot \pi^*\mathrm{A}) = \Psi(\alpha) \cdot f^*\mathrm{A}$ so $\Psi(\alpha) \in \mathrm{N}_{\le 1}(\mathrm{Y})$. On the other hand, we also find $\pi_*(\alpha) = \mathbf{R}f_*\Psi(\alpha) = 0$. But then $\alpha \in \mathrm{N}_0(\mathcal{X}) \cap \mathrm{N}_1(\mathcal{X}) = 0$ and the result follows. $\square$

Now we are in a position to introduce the analogue of slope stability on an orbifold.

Definition 2.1.39. Given $\mathrm{F} \in \mathrm{Coh}_{\le 1}(\mathcal{X})$, the *Nironi slope of* F is

$$\nu(\mathrm{F}) := \frac{a_0(\mathrm{F})}{a_1(\mathrm{F})} \in \mathbf{Q} \cup \{\infty\} \tag{2.1.27}$$

where $\nu(\mathrm{F}) = \infty$ only if $a_1(\mathrm{F}) = 0$. The latter occurs if and only if $\mathrm{F} \in \mathrm{Coh}_0(\mathcal{X})$.

Note that $p_{\mathrm{F}\otimes\mathrm{A}}(t) = p_{\mathrm{F}}(t+1) = a_1(\mathrm{F})t + a_0(\mathrm{F}) + a_1(\mathrm{F})$. Hence, $\nu(\mathrm{F}(k)) = \nu(\mathrm{F}) + k$.

Remark 2.1.40. We emphasize that the notion of slope or Nironi stability on an orbifold depends not only on the choice of an ample line bundle A on X, but also on the choice of generating vector bundle V on $\mathcal{X}$.

Proposition 2.1.41. The slope function ν defines a stability condition on $\mathrm{Coh}_{\le 1}(\mathcal{X})$.

Proof. Since the modified Hilbert polynomial is well-defined on numerical equivalence classes, ν descends to a slope function

$$\nu\colon \mathrm{N}_{\le 1}(\mathcal{X}) \to (\mathbf{Q} \cup \{\infty\}, \le), \tag{2.1.28}$$

where $\mathbf{Q}$ carries its usual ordering and we impose $q \le \infty$ for all $q \in \mathbf{Q}$. Moreover, it is easy to see that ν satisfies the see-saw property.

Finally, the existence of Harder–Narasimhan filtrations follows from [Nir08, Thm. 3.22] for a pure one-dimensional sheaf. For a non-pure sheaf, combine this result with the usual torsion filtration; see [Nir08, Cor. 3.7].

Alternatively, one may argue directly as follows. It suffices to show that $\mathrm{Coh}_{\le 1}(\mathcal{X})$ is ν-Artinian. Let $\mathrm{E} \in \mathrm{Coh}_{\le 1}(\mathcal{X})$, and suppose for a contradiction that we have an infinite sequence of subobjects

$$\mathrm{E} = \mathrm{E}_0 \supset \mathrm{E}_1 \supset \mathrm{E}_2 \supset \cdots$$

such that $\nu(\mathrm{E}_i) \ge \nu(\mathrm{E}_{i+1})$. By Lemma 2.1.38, the set $\{\beta \in \mathrm{N}_1(\mathcal{X}) \mid 0 \le \beta \le \beta_{\mathrm{E}}\}$ is finite, so we reduce to the case $\beta_{\mathrm{E}_i} = \beta_{\mathrm{E}}$ for all i. But then $\mathrm{E}_i \subset \mathrm{E}_{i-1}$ has zero-dimensional cokernel, which implies $\nu(\mathrm{E}_i) < \nu(\mathrm{E}_{i-1})$ by the see-saw property: a contradiction. $\square$

2.1.4 Moduli of Nironi stable sheaves

In [Nir08], F. Nironi constructs a finite type moduli stack of slope semistable coherent sheaves on $\mathcal{X}$ under certain conditions. We collect the relevant results in this section.

Definition 2.1.42. Let $\mathrm{F} \in \mathrm{Coh}_{\leq 1}(\mathcal{X})$. We write $\mathrm{F}_{\max}$ and $\mathrm{F}_{\min}$ for the semistable factors of F with the biggest and smallest slope in its HN filtration respectively.

Remark 2.1.43. We have a natural inclusion $\mathrm{F}_{\max} \hookrightarrow \mathrm{F}$ and surjection $\mathrm{F} \twoheadrightarrow \mathrm{F}_{\min}$, and $\nu_{\max}(\mathrm{F}) := \nu(\mathrm{F}_{\max}) \geq \nu(\mathrm{F}) \geq \nu(\mathrm{F}_{\min}) =: \nu_{\min}(\mathrm{F})$ by the properties of the HN filtration.

Let $\underline{\mathrm{Coh}}_{\mathcal{X}}$ denote the moduli stack of coherent sheaves on $\mathcal{X}$. It is an algebraic stack locally of finite type over $\mathbf{C}$ by [Nir08, Cor. 2.27]. If S is a scheme of finite type, then an S-valued point of $\underline{\mathrm{Coh}}_{\mathcal{X}}$ is given by an S-flat coherent sheaf E on $\mathcal{X} \times \mathrm{S}$.

This stack decomposes into a disjoint union indexed by classes in $\mathrm{N}(\mathcal{X})$.

Lemma 2.1.44. *Suppose S is a connected scheme and F is an S-flat coherent sheaf on $\mathcal{X} \times \mathrm{S}$. For each point $s \in \mathrm{S}(\mathbf{C})$ let $\mathrm{F}_s = \mathrm{F}|_{\mathcal{X} \times \{s\}}$ be the corresponding sheaf on $\mathcal{X}$. Then the class $[\mathrm{F}_s] \in \mathrm{N}(\mathcal{X})$ is independent of the point s.*

Proof. The argument of [Bri12, Lem. 4.5] goes through for $\mathrm{Coh}(\mathcal{X})$. ☐

Corollary 2.1.45. *The stack $\underline{\mathrm{Coh}}_{\mathcal{X}}$ splits up as a disjoint union*

$$\underline{\mathrm{Coh}}_{\mathcal{X}} = \bigsqcup_{\alpha \in \mathrm{N}(\mathcal{X})} \underline{\mathrm{Coh}}_{\mathcal{X},\alpha} \tag{2.1.29}$$

of open and closed substacks, where $\underline{\mathrm{Coh}}_{\mathcal{X},\alpha}$ parametrizes sheaves on $\mathcal{X}$ of class α.

Definition 2.1.46. If $\mathrm{I} \subset (-\infty, \infty]$ is any interval, we let $\underline{\mathcal{M}}^{\mathrm{ss}}_{\nu}(\mathrm{I}) \subset \underline{\mathrm{Coh}}_{\mathcal{X}}$ denote the substack parametrising sheaves F for which $\nu_{\min}(\mathrm{F}), \nu_{\max}(\mathrm{F}) \in \mathrm{I}$.

In other words, the semistable factors of F all have slopes contained in I. When the interval consists of a single point, $\mathrm{I} = [p, p]$, we simply write $\underline{\mathcal{M}}^{\mathrm{ss}}_{\nu}(p)$ instead. For $\beta \in \mathrm{N}_{\leq 1}(\mathcal{X})$, we write $\underline{\mathcal{M}}^{\mathrm{ss}}_{\nu}(\mathrm{I}, \beta)$ for the restriction of $\underline{\mathcal{M}}^{\mathrm{ss}}_{\nu}(\mathrm{I})$ to the open substack

$$\underline{\mathrm{Coh}}_{\mathcal{X},\beta} = \bigsqcup_{c \in \mathrm{N}_0(\mathcal{X})} \underline{\mathrm{Coh}}_{\mathcal{X},\beta+c}. \tag{2.1.30}$$

The following is the key result about these moduli stacks.

Theorem 2.1.47 (Nironi)**.** *If $\mathrm{I} \subseteq \mathbf{R}$ is an interval and $\beta \in \mathrm{N}_1(\mathcal{X})$, then the substack $\underline{\mathcal{M}}^{\mathrm{ss}}_{\nu}(\mathrm{I}, \beta) \subset \underline{\mathrm{Coh}}_{\mathcal{X},\beta}$ is open. If the interval is of finite length, the stack is of finite type.*

In particular, $\underline{\mathcal{M}}^{\mathrm{ss}}_{\nu}(\delta, \beta)$ is of finite type for any $\delta \in \mathbf{R}$.

Proof. These results follow by the Grothendieck lemma for stacks [Nir08, Lem. 4.13], and applying the same proof as in [Nir08, Prop. 4.15] and [HL10, Prop. 2.3.1]. ☐

2.2 The Behrend function and moduli of complexes

Donaldson–Thomas and Pandharipande–Thomas invariants are defined by integrating a certain zero-dimensional cycle, which is associated to a symmetric perfect obstruction theory, over the moduli space. In the fundamental paper [Beh09], K. Behrend proved that, if the moduli space is proper and embeddable, this integral can be expressed as an Euler characteristic weighted with respect to a certain constructible function ν.

K. Behrend's result is crucial for the viability of the motivic Hall algebra approach to results in curve counting; indeed, his result is built into the integration morphism.

First, we briefly discuss the properties we need of Behrend's constructible function. Second, we recall D. Joyce and Y. Song's extension of it to algebraic stacks that are merely *locally* of finite type. Finally, we recall M. Lieblich's *Mother of all moduli*: the moduli stack of gluable objects in the bounded derived category of coherent sheaves. Every instance of 'the' Behrend function we encounter is a restriction of its function.

2.2.1 The Behrend function

We recall some material on constructible functions, which can be found in [JS12].

Definition 2.2.1. Let X be a scheme. A subset $\mathrm{C} \subseteq \mathrm{X}(\mathbf{C})$ is called *constructible* if

$$\mathrm{C} = \bigcup_{i=1}^{n} \mathrm{X}_i(\mathbf{C}), \tag{2.2.1}$$

where $\{\mathrm{X}_i\}$ is a finite collection of finite type subschemes of X. A subset $\mathrm{S} \subseteq \mathrm{X}(\mathbf{C})$ is *locally constructible* if $\mathrm{S} \cap \mathrm{C}$ is constructible for all constructible subsets $\mathrm{C} \subseteq \mathrm{X}(\mathbf{C})$.

Remark 2.2.2. We may take the union in (2.2.1) to be *disjoint*, since all subschemes are locally closed. Moreover, by passing to a finite cover by open affine subschemes, the X_i can be chosen to be *separated*.

The definition of (locally) constructible subsets in an algebraic stack is the same, apart from replacing 'scheme' by 'stack' everywhere. We have the following notion.

Definition 2.2.3. Let X be an algebraic stack. A function $f \colon \mathrm{X}(\mathbf{C}) \to \mathbf{Q}$ is called *constructible on* X if

1. the set $f(\mathrm{X}(\mathbf{C}))$ is finite, and
2. $f^{-1}(c)$ is a constructible subset of $\mathrm{X}(\mathbf{C})$ for all $c \in f(\mathrm{X}(\mathbf{C}))\backslash\{0\}$.

The function f is called *locally constructible on* X if $f \cdot \delta_{\mathrm{C}}$ is constructible for every constructible subset C of $\mathrm{X}(\mathbf{C})$, where δ_{C} is the characteristic function of the subset C.

We write $\mathrm{C}(\mathrm{X})$ for abelian group of constructible functions on X.

The idea is that for X not of finite type, $X(\mathbf{C})$ can be very 'large' or 'unbounded'. Constructible functions are only non-zero on small, bounded subsets of $X(\mathbf{C})$, and $f^{-1}(0)$ is the large, unbounded remainder. They behave nicely with respect to union, intersection, and complement, because constructible functions do.

We state the existence and properties of Behrend's function as a Theorem.

Theorem 2.2.4. Let X be a finite type scheme. There exists a canonical constructible function

$$\nu_X : X(\mathbf{C}) \longrightarrow \mathbf{Z},$$

with the following properties, where Y is another finite type scheme:

1. At smooth points p of X we have $\nu_X(p) = (-1)^{\dim X}$.
2. If $f: X \to Y$ is a smooth morphism of relative dimension d, then $f^*\nu_Y = (-1)^d \nu_X$.
3. In particular, if $f: X \to Y$ is an étale morphism, then $f^*\nu_Y = \nu_X$; hence, $\nu_X(p)$ is an invariant of the singularity of X at p.
4. Multiplicativity: $\nu_{X\times Y} = \nu_X \boxdot \nu_Y$, where $(\nu_X \boxdot \nu_Y)(p, q) := \nu_X(p)\nu_Y(q)$.
5. If X is the critical locus of a regular function $f: M \to \mathbf{A}^1$ on a smooth scheme M, i.e., $X = Z(df)$, then

$$\nu_X(p) = (-1)^{\dim M}(1 - e(M_p(f))), \tag{2.2.2}$$

   where F_p is the *Milnor fibre*, i.e., the intersection of a nearby fibre of f with a small ball in M centred at p, and e denotes the topological Euler characteristic.

Typically, it is notoriously difficult to determine the Behrend function of a given scheme, unless the scheme is smooth or particular, as illustrated by the following.

Example 2.2.5. Let $X_n = \operatorname{Spec} \mathbf{C}[t]/(t^n)$ be the fat point for $n \in \mathbf{Z}_{\geq 1}$. Consider the regular function $f_n: \mathbf{A}^1 \to \mathbf{A}^1$ given by $f_n(t) = t^{n+1}$. Clearly $df_n = (n+1)t^n\, dt$ so that

$$X_n \equiv \operatorname{Spec} \mathbf{C}[t]/(t^n) = Z(df_n) \tag{2.2.3}$$

arises as the critical locus of a regular function. The Milnor fibre of f_n at $p \in \mathbf{A}^1$ is

$$M_p(f_n) := \{x \in \mathbf{A}^1 \,:\, \|x - p\| < \delta,\, f_n(x) = f_n(p) + \epsilon,\, 0 < \epsilon \ll \delta \ll 1\} \tag{2.2.4}$$

where $\| - \|$ denotes the usual norm on $\mathbf{A}^1 = \mathbf{C}$. In this case, the fibre of f_n over a point $x \in \mathbf{A}^1$ nearby to 0 contains $n+1$ points. One deduces that $e(M_0(f_n)) = n+1$. Since $\dim \mathbf{A}^1 = 1$ we conclude that $\nu_{X_n} = n$.

This exhibits the sensitivity of ν_X to the singularity structure of X.

D. Joyce and Y. Song's extend the construction and properties of the Behrend function to algebraic stacks that are merely *locally* of finite type.

Proposition 2.2.6. Let X be an algebraic stack, then there exists a canonical locally constructible function $\nu_X : X(\mathbf{C}) \to \mathbf{Z}$ that is uniquely characterised by the following property. If W is a finite type scheme and $f : W \to X$ is a smooth morphism of relative dimension d, then $f^*\nu_X = (-1)^d \nu_W$ as constructible functions on W.

Properties 1. through 4. of Theorem 2.2.4 then also hold for ν_X.

Proof. This is [JS12, Prop. 4.4]. □

The Behrend function of a scheme or stack should be thought of as a weight function, encoding properties of the singularity structure of the space. The corresponding weighted Euler characteristic plays a key role in the theory of counting invariants.

Definition 2.2.7. The Behrend weighted Euler characteristic of a proper scheme X is

$$e_B(X) := e(X, \nu_X) := \sum_{k \in \mathbf{Z}} k e(\nu_X^{-1}(k)) \in \mathbf{Z}. \tag{2.2.5}$$

The following is K. Behrend's fundamental theorem about his constructible function.

Theorem 2.2.8. Let X be a proper scheme equipped with a symmetric perfect obstruction theory. Then the virtual count of points in X is

$$\int_{[X]^{\mathrm{vir}}} 1 = e_B(X) \tag{2.2.6}$$

where $[X]^{\mathrm{vir}}$ denote the associated virtual fundamental class

Proof. This is [Beh09, Thm. 4.18]. □

In particular, this result shows that the virtual count of points in a proper scheme does not depend on the symmetric perfect obstruction theory chosen. Moreover, it shows that Donaldson–Thomas and Pandharipande–Thomas invariants may be defined and computed using motivic cut-and-paste techniques allowed by the topological Euler characteristic e and the constructible function of Behrend.

2.2.2 The mother of all moduli

Curves are counted by taking the Behrend weighted Euler characteristic of their moduli space. Thus, in order to relate the various counts, we need to both relate their moduli spaces and the corresponding Behrend functions.

We construct these moduli stacks as open substacks of M. Lieblich's moduli stack of gluable objects in the derived category, constructed in [Lie06]. It is an algebraic stack locally of finite type over $\mathbf{C}$. Part 3 of Theorem 2.2.4 then allows us to relate the corresponding Behrend functions by simply restricting over the open inclusion.

Here we collect the various facts we need about this *mother of all moduli* in our setting. So let Y denote a smooth proper variety, with structure morphism $\pi\colon \mathrm{Y} \to \mathrm{Spec}(\mathbf{C})$ which is then proper, flat (even smooth), and of finite presentation.

Definition 2.2.9. Let $\mathfrak{Mum}_{\mathrm{Y}}$ denote the pre-stack that associates to a scheme T the groupoid of objects E in $\mathrm{D}(\mathrm{Y}_{\mathrm{T}})$ which are T-perfect and such that $\mathrm{Ext}^i_{\mathrm{Y}_t}(\mathrm{E}_t, \mathrm{E}_t) = 0$ for all geometric points $t \to \mathrm{T}$ and all $i < 0$. Here $\mathrm{Y}_{\mathrm{T}} = \mathrm{T} \times_{\mathbf{C}} \mathrm{Y}$ is the base-change and $\mathrm{E}_t = \mathbf{L}i_t^*(\mathrm{E})$ denotes the derived restriction of E over $i_t\colon \mathrm{Y}_t \to \mathrm{Y}_{\mathrm{T}}$.

Remark 2.2.10. The condition that E be T-perfect means that there exists an open cover $\{\mathrm{U}_i\}$ of T such that $\mathrm{E}|_{\mathrm{U}_i}$ is quasi-isomorphic to a bounded complex of quasi-coherent sheaves flat over T. Equivalently, E is T-perfect if and only if $\mathrm{E}_t \in \mathrm{D}(\mathrm{Y})$ for all $t \in \mathrm{T}$. As $\mathrm{Y} \times \mathrm{T} \to \mathrm{T}$ is smooth E is T-perfect if and only if $\mathrm{E} \in \mathrm{Perf}(\mathrm{T} \times \mathrm{Y})$ is perfect.

Remark 2.2.11. By construction, any object E in the heart of a bounded t-structure on $\mathrm{D}(\mathrm{Y})$ has no negative self-extensions $\mathrm{Ext}^i_{\mathrm{Y}}(\mathrm{E}, \mathrm{E}) = 0$ for $i < 0$. In particular, any such object determines a $\mathbf{C}$-valued point of $\mathfrak{Mum}_{\mathrm{Y}}$.

The following is the key result of [Lie06] in our setting.

Theorem 2.2.12. $\mathfrak{Mum}_{\mathrm{Y}}$ is an algebraic stack that is locally of finite type over $\mathbf{C}$.

Proof. This is [Lie06, Thm. 4.2.1]. □

Corollary 2.2.13. There exists a canonical locally constructible function on $\mathfrak{Mum}_{\mathrm{Y}}$, denoted $\nu_{\mathfrak{M}}\colon \mathfrak{Mum}_{\mathrm{Y}}(\mathbf{C}) \to \mathbf{Z}$, satisfying properties 1. through 4. of Theorem 2.2.4.

We refer to this function as *the Behrend function of* $\mathfrak{Mum}_{\mathrm{Y}}$.

Proof. This follows from Proposition 2.2.6. □

The stack $\mathfrak{Mum}_{\mathrm{Y}}$ splits up as a disjoint union of open and closed substacks

$$\mathfrak{Mum}_{\mathrm{Y}} = \coprod_{\alpha \in \mathrm{N}(\mathrm{Y})} \mathfrak{Mum}_{\mathrm{Y},\alpha} \tag{2.2.7}$$

where $\mathfrak{Mum}_{\mathrm{Y},\alpha}$ parametrises objects of class $\alpha \in \mathrm{N}(\mathrm{Y})$.

Lemma 2.2.14. For any interval $[a, b]$, there is an open substack $\mathfrak{Mum}_{\mathrm{Y}}^{[a,b]} \subset \mathfrak{Mum}_{\mathrm{Y}}$, parametrising complexes $\mathrm{E} \in \mathrm{D}(\mathrm{Y})$ concentrated in cohomological degrees $[a, b]$.

Proof. This is for example explained in [Cal16a, App. A]. $\square$

Remark 2.2.15. As a special case, the stack of coherent sheaves

$$i\colon \underline{\mathrm{Coh}}_{\mathrm{Y}} = \mathfrak{Mum}_{\mathrm{Y}}^{[0,0]} \subset \mathfrak{Mum}_{\mathrm{Y}}$$

is an open substack. As an algebraic stack locally of finite type it carries a Behrend function ν_{Y}. By openness of i, these Behrend functions are compatible $i^*\nu_{\mathfrak{M}} = \nu_{\mathrm{Y}}$.

Remark 2.2.16. Given a subcategory $\mathtt{C} \subset \mathrm{D}(\mathrm{Y})$ whose objects have vanishing negative self-extensions, our general convention is to denote $\underline{\mathtt{C}} \subset \mathfrak{Mum}_{\mathrm{Y}}$ the corresponding substack. Recall that, given a finite type $\mathbf{C}$-scheme T, its T-valued points are

$$\underline{\mathtt{C}}(\mathrm{T}) = \{\mathrm{E} \in \mathfrak{Mum}_{\mathrm{Y}}(\mathrm{T}) \mid \mathrm{E}_t \in \mathtt{C}, \forall\, t \in \mathrm{T}\}$$

where $t \in \mathrm{T}$ is a geometric point and $\mathrm{E}_t = \mathbf{L}i_t^*\mathrm{E}$ is the derived fibre in the diagram

$$\begin{tikzcd} \mathrm{Y}_t \arrow[r, "i_t"] \arrow[d, "\pi_t"'] & \mathrm{Y}_{\mathrm{T}} \arrow[r] \arrow[d, "\pi_{\mathrm{T}}"] & \mathrm{Y} \arrow[d, "\pi"] \\ t \arrow[r, "i"] & \mathrm{T} \arrow[r] & \mathrm{Spec}(\mathbf{C}) \end{tikzcd} \tag{2.2.8}$$

where both squares are Cartesian.

Remark 2.2.17. To be precise, one should look at all scheme-theoretic points $t \in \mathrm{T}$, and not just the geometric ones. If K denotes a field extension of $\mathbf{C}$ (for example the residue field of $t \in \mathrm{T}$), we implicitly assume there exists an natural analogue $\mathtt{C}_{\mathrm{K}}$ of the category $\mathtt{C}$ over K, and that checking the membership $\mathrm{E} \in \mathtt{C}_{\mathrm{K}}$ may be done by base changing to the algebraic closure $\mathtt{C}_{\overline{\mathrm{K}}}$.

For the applications we have in mind, namely substacks defined by *open torsion pairs* as in Definition 4.2.9, this assumption is fully justified by [AB13, Lem. A.1].

Constructing a stack as a substack of $\mathfrak{Mum}_{\mathrm{Y}}$ has the following upshot.

Corollary 2.2.18. If $\mathtt{C} \subset \mathrm{D}(\mathrm{Y})$ is a subcategory such that the corresponding moduli stack $i\colon \underline{\mathtt{C}} \subset \mathfrak{Mum}_{\mathrm{Y}}$ is open, then

1. $\underline{\mathtt{C}}$ is an algebraic stack locally of finite type over $\mathbf{C}$, and
2. it has a Behrend function $\nu_{\mathtt{C}}$ that satisfies $i^*\nu_{\mathfrak{M}} = \nu_{\mathtt{C}}$.

This is a direct generalisation of Remark 2.2.15.

Proof. The first statement is immediate by transfer of structure, whereas the second follows from property 3. of Theorem 2.2.4. $\square$

2.3 The motivic Hall algebra

We recall the construction of D. Joyce's motivic Hall algebra associated to an abelian category following the approach of T. Bridgeland [Bri12]. In principle, this construction works for any abelian category whose objects are parametrised by an algebraic stack that is locally of finite type, and such that relative Quot functors are representable by finite type schemes.

However, we make no attempt at maximal generality here and focus on the case at hand: the heart of a bounded t-structure $\mathtt{C} \subset \mathrm{D}(\mathcal{X})$ where $\mathcal{X}$ is a smooth projective CY3 orbifold. For a more detailed discussion about general aspects of the theory, we refer to [Bri11], for the CY3 setting we refer to [Bri12] and [Tod16a].

2.3.1 Grothendieck groups of stacks

Let $\mathrm{Var}/\mathbf{C}$ denote the category of varieties. The Grothendieck ring of varieties is the universal ring satisfying the so-called cut-and-paste or motivic relations.

Definition 2.3.1. Let $\mathrm{K}(\mathrm{Var}/\mathbf{C})$ denote the $\mathbf{Q}$-vector space generated by isomorphisms classes of varieties $[\mathrm{X}]$, modulo the scissor relations

$$[\mathrm{X}] = [\mathrm{Z}] + [\mathrm{U}] \tag{2.3.1}$$

for $\mathrm{Z} \subset \mathrm{X}$ a closed subvariety and $\mathrm{U} = \mathrm{X} \backslash \mathrm{Z}$ the complementary open subvariety.

There is a natural structure of commutative ring on $\mathrm{K}(\mathrm{Var}/\mathbf{C})$ given by setting $[\mathrm{X}] \cdot [\mathrm{Y}] := [\mathrm{X} \times \mathrm{Y}]$, and the class of a point $1 = [\mathrm{Spec}(\mathbf{C})]$ is the unit. An example of a *motivic* invariant, one respecting the scissor relations, is the Euler characteristic.

Example 2.3.2. There is a natural ring homomorphism $e\colon \mathrm{K}(\mathrm{Var}/\mathbf{C}) \to \mathbf{Z}$ defined by sending a complex variety X to its topological Euler characteristic $e(\mathrm{X}) \in \mathbf{Z}$. Indeed $e(\mathrm{X}) = e(\mathrm{Z}) + e(\mathrm{U})$ if equation (2.3.1) holds, and we have $e(\mathrm{X} \times \mathrm{Y}) = e(\mathrm{X})e(\mathrm{Y})$.

The class of a variety in $\mathrm{K}(\mathrm{Var}/\mathbf{C})$ is insensitive to passing to a *stratification*, i.e., a collection of disjoint locally-closed subsets which together cover the variety.

Lemma 2.3.3. If a variety X is stratified by subvarieties X_i, then only finitely many of X_i are non-empty and

$$[\mathrm{X}] = \sum_i [\mathrm{X}_i] \in \mathrm{K}(\mathrm{Var}/\mathbf{C}). \tag{2.3.2}$$

Proof. This is [Bri12, Lem. 2.2]. □

In order to extend the definition of Grothendieck rings to categories of schemes or stacks possibly of infinite type, it is useful to reformulate the definition of the Grothendieck ring. Recall that a scheme is of finite type unless otherwise stated.

Definition 2.3.4. A morphism $f\colon \mathrm{X} \to \mathrm{Y}$ of schemes is called a *geometric bijection* if it induces a bijection $f(\mathbf{C})\colon \mathrm{X}(\mathbf{C}) \to \mathrm{Y}(\mathbf{C})$ between the sets of $\mathbf{C}$-valued points.

The alternative description of the Grothendieck group is then as follows.

Lemma 2.3.5. The group $\mathrm{K}(\mathrm{Var}/\mathbf{C})$ is the $\mathbf{Q}$-vector space generated by isomorphism classes of varieties, modulo the relations

1. $[\mathrm{X} \sqcup \mathrm{Y}] = [\mathrm{X}] \sqcup [\mathrm{Y}]$ for every pair of varieties X and Y,
2. $[\mathrm{X}] = [\mathrm{Y}]$ for every geometric bijection $f\colon \mathrm{X} \to \mathrm{Y}$.

Proof. This is [Bri12, Lem. 2.9]. ☐

Let $\mathrm{Sch}/\mathbf{C} \subset \mathrm{Sp}/\mathbf{C}$ denote the categories of schemes and algebraic spaces respectively, of finite type. We define Grothendieck rings of schemes $\mathrm{K}(\mathrm{Sch}/\mathbf{C})$ and algebraic spaces $\mathrm{K}(\mathrm{Sp}/\mathbf{C})$ as the $\mathbf{Q}$-vector space generated by isomorphism classes in said category modulo the relations in the above lemma. In doing so, we obtain nothing new.

Lemma 2.3.6. The embeddings of categories $\mathrm{Var}/\mathbf{C} \subset \mathrm{Sch}/\mathbf{C} \subset \mathrm{Sp}/\mathbf{C}$ induce isomorphisms of rings $\mathrm{K}(\mathrm{Var}/\mathbf{C}) \cong \mathrm{K}(\mathrm{Sch}/C) \cong \mathrm{K}(\mathrm{Sp}/\mathbf{C})$. In particular, $[\mathrm{X}] = [\mathrm{X}_{\mathrm{red}}]$ for $\mathrm{X} \in \mathrm{Sch}/\mathbf{C}$.

Analogously, we may define a Grothendieck ring of finite type stacks. In order to have a useful comparison result of this ring with the Grothendieck ring of varieties (schemes, algebraic spaces) of the previous section, we restrict the class of stacks.

Definition 2.3.7. An algebraic stack X locally of finite type is said to have *affine geometric stabilisers* if for every $\mathbf{C}$-valued point $x \in \mathrm{X}(\mathbf{C})$ its group of isomorphisms $\mathrm{Isom}_{\mathbf{C}}(x, x)$ is an affine algebraic group.

Example 2.3.8. The general linear group $\mathrm{GL}_n(\mathbf{C})$ is the complement of a global equation $\det = 0$ in affine space $\mathrm{End}(\mathbf{A}^n)$, and as such is an affine algebraic group.

Lemma 2.3.9. Let $\underline{\mathtt{C}} \subset \mathfrak{Mum}_{\mathcal{X}}$ denote the moduli stack parametrising objects in the heart $\mathtt{C}$ of a bounded t-structure on $\mathrm{D}(\mathcal{X})$, and assume the inclusion is open. Then $\underline{\mathtt{C}}$ is an algebraic stack, and it has affine geometric stabilisers.

Proof. Since $\mathcal{X}$ is proper over $\mathbf{C}$ the endomorphism ring $\mathrm{A} := \mathrm{End}(\mathrm{E})$ of any $\mathbf{C}$-valued point $\mathrm{E} \in \underline{\mathtt{C}}(\mathbf{C})$ is a finite-dimensional algebra. Thus, we may choose an isomorphism

$A \cong \mathbf{C}^k$ as vector spaces. Letting A act on itself by composition of endomorphisms induces a closed linear embedding $\phi\colon A \hookrightarrow \mathrm{End}(\mathbf{C}^k)$. Since $a \in A$ is an isomorphism if and only if ϕ_a acts bijectively on A, it follows that $\mathrm{Aut}(E) = \phi^{-1}(\mathrm{GL}_k(\mathbf{C}))$. Thus $\mathrm{Aut}(E) \subset \mathrm{End}(E)$ is an affine algebraic group. □

The importance of this property is the following result by A. Kresch.

Proposition 2.3.10. A finite type algebraic stack X has affine geometric stabilisers if and only if there exist a variety Y with an action of $G = \mathrm{GL}_d$ and a geometric bijection

$$f\colon [Y/G] \to X. \tag{2.3.3}$$

Proof. This is [Bri12, Prop. 3.5] □

Let $\mathrm{St}/\mathbf{C}$ denote the (2-)category of algebraic stacks of finite type.

Definition 2.3.11. The Grothendieck ring $K(\mathrm{St}/\mathbf{C})$ is the $\mathbf{Q}$-vector space generated by symbols $[X]$, where X is a finite type algebraic stack over $\mathbf{C}$ with affine geometric stabilisers, modulo the relations

1. $[X \sqcup Y] = [X] + [Y]$ for every pair of stacks X and Y,
2. If $f\colon X \to Y$ is a geometric bijection, i.e., f induces an equivalence of groupoids $X(\mathbf{C}) \to Y(\mathbf{C})$, then $[X] = [Y]$.
3. If $X_1, X_2 \to Y$ are Zariski fibrations[4] with the same fibres, then $[X_1] = [X_2]$.

As before, taking products $[X]\cdot[Y] := [X \times Y]$ induces a natural structure of commutative ring (in fact, $\mathbf{Q}$-algebra) with unit $\mathbf{1} = [\mathrm{Spec}(\mathbf{C})]$. There is a natural homomorphism of $\mathbf{Q}$-algebras

$$Q'\colon K(\mathrm{Var}/\mathbf{C}) \to K(\mathrm{St}/\mathbf{C}) \tag{2.3.4}$$

obtained by considering a variety as a representable stack.

The result of A. Kresch shows the following comparison result.

Lemma 2.3.12. The morphism (2.3.4) induces an isomorphism of commutative $\mathbf{Q}$-algebras

$$Q\colon K(\mathrm{Var}/\mathbf{C})\left[[\mathrm{GL}_d]^{-1}\colon d \geq 1\right] \to K(\mathrm{St}/\mathbf{C}). \tag{2.3.5}$$

Proof. This is [Bri12, Lem. 3.9]. A key part of the proof is showing that *special* groups such as GL_d, define invertible classes in $K(\mathrm{St}/\mathbf{C})$. As a consequence, if X is a stack with geometric bijection $f\colon [Y/\mathrm{GL}_d] \to X$ as in Proposition 2.3.10, then $Q^{-1}([X]) = [Y]/[\mathrm{GL}_d]$ is well-defined. □

[4]See [Bri12] for the definition of this term, which will not be used in this paper.

Let S be a fixed algebraic stack *locally* of finite type with affine geometric stabilisers. There is a relative version of $\mathrm{K}(\mathrm{St}/\mathbf{C})$.

Definition 2.3.13. The S-relative Grothendieck group of stacks is the $\mathbf{Q}$-vector space $\mathrm{K}(\mathrm{St}/\mathrm{S})$ generated by symbols $[\mathrm{X} \to \mathrm{S}]$, where X is a *finite* type algebraic stack over $\mathbf{C}$ with affine geometric stabilisers, modulo relations

1. For every pair of S-stacks $f_1, f_2 \in \mathrm{St}/\mathrm{S}$, we have

$$[\mathrm{X}_1 \sqcup \mathrm{X}_2 \xrightarrow{f_1 \sqcup f_2} \mathrm{S}] = [\mathrm{X}_1 \xrightarrow{f_1} \mathrm{S}] + [\mathrm{X}_2 \xrightarrow{f_2} \mathrm{S}]. \tag{2.3.6}$$

2. For every geometric bijection $g\colon \mathrm{X}_1 \to \mathrm{X}_2$, we have

$$[\mathrm{X}_1 \xrightarrow{f_1} \mathrm{S}] = [\mathrm{X}_2 \xrightarrow{f_2} \mathrm{S}]. \tag{2.3.7}$$

3. For every pair of Zariski locally trivial fibrations $f_i\colon \mathrm{X}_i \to \mathrm{Y}$ with the same fibres and every morphism $g\colon \mathrm{Y} \to \mathrm{S}$, we have

$$[\mathrm{X}_1 \xrightarrow{g\circ f_1} \mathrm{S}] = [\mathrm{X}_2 \xrightarrow{g\circ f_2} \mathrm{S}]. \tag{2.3.8}$$

The vector space $\mathrm{K}(\mathrm{St}/\mathrm{S})$ is naturally a $\mathrm{K}(\mathrm{St}/\mathbf{C})$-module, where the module structure is given by setting $[\mathrm{X}]\cdot[\mathrm{Y} \to \mathrm{S}] := [\mathrm{X} \times \mathrm{Y} \to \mathrm{Y} \to \mathrm{S}]$. Fibred product over S induces a natural commutative ring structure $[\mathrm{Y} \to \mathrm{S}]\cdot[\mathrm{Z} \to \mathrm{S}] := [\mathrm{Y} \times_{\mathrm{S}} \mathrm{Z} \to \mathrm{S}]$.

Remark 2.3.14. Given a morphism of stacks $a\colon \mathrm{S} \to \mathrm{T}$, there are natural induced morphisms of $\mathrm{K}(\mathrm{St}/\mathbf{C})$-modules between their relative Grothendieck rings of stacks, together with various compatibilities; these are easy to check, see [Bri12, §3.5] for details.

1. Pushforward $a_*\colon \mathrm{K}(\mathrm{St}/\mathrm{S}) \to \mathrm{K}(\mathrm{St}/\mathrm{T})$ via $a_*[\mathrm{X} \xrightarrow{f} \mathrm{S}] = [\mathrm{X} \xrightarrow{a\circ f} \mathrm{T}]$.

2. Pullback $a^*\colon \mathrm{K}(\mathrm{St}/\mathrm{T}) \to \mathrm{K}(\mathrm{St}/\mathrm{S})$ sending $a^*[\mathrm{Y} \xrightarrow{g} \mathrm{T}] = [\mathrm{X} \xrightarrow{f} \mathrm{S}]$ where f is the map appearing in the Cartesian diagram

$$\begin{tikzcd} \mathrm{X} \arrow[r] \arrow[d, "f"'] & \mathrm{Y} \arrow[d, "g"] \\ \mathrm{S} \arrow[r, "a"] & \mathrm{T} \end{tikzcd} \tag{2.3.9}$$

provided that a is of finite type. Indeed, this guarantees that X is of finite type over $\mathbf{C}$ provided that Y is.

3. Pushforward and pullback are functorial.

4. Pushforward and pullback satisfy base-change along a Cartesian diagram.

5. For every pair of stacks S_1, S_2 there is a Künneth map of $K(\mathrm{St}/\mathbf{C})$-modules

$$K\colon K(\mathrm{St}/S_1) \otimes K(\mathrm{St}/S_2) \to K(\mathrm{St}/S_1 \times S_2) \tag{2.3.10}$$

   given by $K([X_1 \to f_1 S_1] \otimes [X_1 \to f_1 S_1]) = [X_1 \times X_2 \to f_1 \times f_2 S_1 \times S_2]$

2.3.2 An algebra structure via extensions

We describe the algebra structure on the Hall algebra in our setting. Thus, let $\mathcal{X}$ be a (quasi-)projective CY3 orbifold as in Definition 2.1.21. In Corollary 4.2.3, we establish the existence of an analogous algebraic stack $\mathfrak{Mum}_{\mathcal{X}}$ that is locally of finite type.

Let $\mathtt{C} \subset D(\mathcal{X})$ be the heart of a bounded t-structure. We assume that the moduli stack $\underline{\mathtt{C}} \subset \mathfrak{Mum}_{\mathcal{X}}$ forms an open substack, locally of finite type, with affine geometric stabilisers. The reader can think of $\mathtt{C} = \mathrm{Coh}(\mathcal{X})$; openness is proven in Lemma 2.2.14.

Central to defining the product on the motivic Hall algebra are short exact sequences. There exists a stack $\mathfrak{Ex}$ of short exact sequences in the category $\mathtt{C}$. It comes with three distinguished maps $\pi_i\colon \mathfrak{Ex} \to \underline{\mathtt{C}}$, $i = 1, 2, 3$. The map π_i corresponds to sending a short exact sequence $0 \to E_1 \to E_2 \to E_3 \to 0$ to the object E_i.

Proposition 2.3.15. The stack $\mathfrak{Ex}$ is an algebraic stack, locally of finite type. The morphism $(\pi_1, \pi_3)\colon \mathfrak{Ex} \to \underline{\mathtt{C}} \times \underline{\mathtt{C}}$ is of finite type.

Sketch. This follows by [Bri12, Lem. 4.1 & 4.2], provided that the relative Quot-functor for $\mathtt{C}$ is represented by an algebraic space. Given M. Lieblich's result about $\mathfrak{Mum}_{\mathcal{X}}$, the proof of the representability of the usual Quot-functor given in the Stacks Project [Sta17, Tag 09TQ] should go through with small modifications. □

Remark 2.3.16. We are only concerned with the Hall algebra associated to the heart $\mathtt{C} = \langle \mathrm{Coh}_{\geq 2}(\mathcal{X})[1], \mathrm{Coh}_{\leq 1}(\mathcal{X})\rangle$ introduced in Definition 4.1.2. It was pointed out to us by J. Calabrese that the results of [Low11, Prop. 6.1] are sufficient to prove the claim for this heart. Indeed, there is a stack $\mathfrak{Ar}^{[-1,0]}$ parametrising arrows in $D^{[-1,0]}(\mathcal{X})$. This is the classical truncation of [Low11, Prop. 5.10], hence it is algebraic and locally of finite type. Our stack $\mathfrak{Ex}$ is obtained as a fibre product $\mathfrak{Ex} = \mathfrak{Ar}^{[-1,0]} \times_{\mathfrak{Mum}_{\mathcal{X}} \times \mathfrak{Mum}_{\mathcal{X}}} (\underline{\mathtt{C}} \times \underline{\mathtt{C}})$.

Note that the results of [Low11] are valid for projective schemes, but using the dg equivalence $\Phi\colon D(Y) \to D(\mathcal{X})$ of the McKay equivalence of Theorem 2.4.11. the same results are valid for $D(\mathcal{X})$.

Pulling back by $(\pi_1, \pi_3)\colon \mathfrak{Ex} \to \underline{\mathtt{C}} \times \underline{\mathtt{C}}$ and pushing forward over $\pi_2\colon \mathfrak{Ex} \to \underline{\mathtt{C}}$ should be thought of as taking the universal extension; heuristically, we think of $\mathfrak{Ex}$ as the product of the $\underline{\mathtt{C}}$-stack $\mathbf{1}\colon \underline{\mathtt{C}} \to \underline{\mathtt{C}}$ with itself as discussed in [ST11]. The product of any two other $\underline{\mathtt{C}}$-stacks is formed by the fibred product.

Given two elements $[X_1 \to \underline{\mathtt{C}}]$, $[X_2 \to \underline{\mathtt{C}}]$ of $K(\mathrm{St}/\underline{\mathtt{C}})$, take their fibred product

$$\begin{tikzcd} X_1 * X_2 \arrow[r, "a"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{Ex} \arrow[r, "\pi_2"] \arrow[d, "(\pi_1,\pi_3)"] & \underline{\mathtt{C}} \\ X_1 \times X_2 \arrow[r] & \underline{\mathtt{C}} \times \underline{\mathtt{C}} & \end{tikzcd} \tag{2.3.11}$$

Note that $X_1 * X_2$ is again a finite-type stack over $\mathbf{C}$ with affine geometric stabilisers. Thus the class of the top horizontal line of the diagram $[\pi_2 \circ a\colon X_1 * X_2 \to \underline{\mathtt{C}}]$ defines an element of $K(\mathrm{St}/\underline{\mathtt{C}})$.

Proposition 2.3.17. The operation $([X_1 \to \underline{\mathtt{C}}], [X_2 \to \underline{\mathtt{C}}]) \mapsto [X_1 * X_2 \to \underline{\mathtt{C}}]$ defines an associative product on $K(\mathrm{St}/\underline{\mathtt{C}})$ with unit $\mathbf{1}_0 = [\underline{\mathtt{C}}_0 \subset \underline{\mathtt{C}}]$ corresponding to the stack of zero objects in $\mathtt{C}$.

Proof. The proof is identical to [Bri12, Thm. 4.3], as it essentially follows from the fact that the first isomorphism theorem holds in any abelian category, such as $\mathtt{C}$. □

Definition 2.3.18. The *motivic Hall algebra* of $\mathtt{C}$ is $H(\mathtt{C}) := (K(\mathrm{St}/\underline{\mathtt{C}}), *, \mathbf{1}_0)$.

Remark 2.3.19. We can express the multiplication $m\colon H(\mathtt{C}) \otimes H(\mathtt{C}) \to H(\mathtt{C})$ explicitly in terms of the functorialities of Remark 2.3.14. In those terms, it is given as

$$m = \pi_{2,*} \circ (\pi_1, \pi_3)^* \circ K, \tag{2.3.12}$$

where pullback by (π_1, π_3) is well-defined since this morphism is of finite type.

There is some additional structure on the Hall algebra. First, as for any relative Grothendieck ring of stacks, taking Cartesian products turns $H(\mathtt{C})$ into an algebra over $K(\mathrm{St}/\mathbf{C})$. Second, elements of the Hall algebra are naturally graded by the numerical Grothendieck group $N(\mathcal{X})$.

Definition 2.3.20. An element $[f\colon X \to \underline{\mathtt{C}}]$ is *homogeneous of degree* α if f factors through the substack $\underline{\mathtt{C}}_\alpha \subset \underline{\mathtt{C}}$, where $\underline{\mathtt{C}}_\alpha = \underline{\mathtt{C}} \cap \mathfrak{Num}_{\mathcal{X},\alpha}$ induced by (2.2.7) (or (4.2.13)).

The inclusion $\underline{\mathtt{C}}_\alpha \subset \underline{\mathtt{C}}$ induces an embedding $K(\mathrm{St}/\mathtt{C}_\alpha) \subset K(\mathrm{St}/\mathtt{C})$, and hence a direct sum decomposition

$$H(\mathtt{C}) = \bigoplus_{\alpha \in N(\mathcal{X})} H(\mathtt{C})_\alpha. \tag{2.3.13}$$

This turns $H(\mathtt{C})$ into a $N(\mathcal{X})$-graded algebra since the product is defined by taking extensions whose classes in $N(\mathcal{X})$ are, by construction, compatible. Moreover, the $K(\mathrm{St}/\mathbf{C})$-algebra structure respects this grading.

Corollary 2.3.21. The motivic Hall algebra $\mathrm{H}(\mathtt{C})$ is an associative and unital $\mathrm{N}(\mathcal{X})$-graded $\mathrm{K}(\mathrm{St}/\mathbf{C})$-algebra.

And third, there is a Poisson bracket on $\mathrm{H}(\mathtt{C})$ given by the formula

$$\{f, g\} = \frac{f * g - g * f}{\mathbf{L} - 1}. \tag{2.3.14}$$

Note that the element $[\mathbf{C}^\times] = \mathbf{L} - 1$ is invertible in $\mathrm{K}(\mathrm{St}/\mathbf{C})$, and hence is indeed invertible in $\mathrm{H}(\mathtt{C})$, since $[\mathrm{B}\mathbf{C}^\times][\mathbf{C}^\times] = \mathbf{1}_0$.

2.3.3 Integration map

Because $\mathrm{D}(\mathcal{X})$ is a CY3 triangulated category, i.e., Serre duality acts as $[3]$, a certain subquotient of the Hall algebra $\mathrm{H}(\mathtt{C})$ supports a Poisson algebra morphism into a formal power series ring. This so-called *integration morphism* is key in translating categorical identities in the Hall algebra into equalities of generating series.

We first introduce the correct subquotient of $\mathrm{H}(\mathtt{C})$, which is the domain of the integration morphism.

Definition 2.3.22. The subalgebra of *regular elements* $\mathrm{H}_{\mathrm{reg}}(\mathtt{C}) \subset \mathrm{H}(\mathtt{C})$ is the $\mathrm{K}(\mathrm{Var}/\mathbf{C})[\mathbf{L}^{-1}, \{[\mathbf{P}^n]^{-1}\}_{n\geq 1}]$-module generated by those elements of $\mathrm{H}(\mathtt{C})$ that are of the form $[\mathrm{Y} \to \underline{\mathtt{C}}]$ where Y is a variety. Here, $\mathbf{L} = [\mathbf{A}^1_{\mathbf{C}}]$ denotes the *Lefschetz motive*.

Remark 2.3.23. The classes $[\mathbf{P}^n]$ for $n \geq 1$ are not inverted in the published version of the article [Bri11]. However, Y. Toda pointed out the necessity of inverting these classes as explained in the corrected version [Bri10].

The following result shows that $\mathrm{H}_{\mathrm{reg}}(\mathtt{C})$ is closed under the product $*$.

Proposition 2.3.24. The subspace $\mathrm{H}_{\mathrm{reg}}(\mathtt{C})$ is closed under the Hall algebra product and is therefore a $\mathrm{K}(\mathrm{Var}/\mathbf{C})[\mathbf{L}^{-1}, \{[\mathbf{P}^n]^{-1}\}_{n\geq 0}]$-algebra.

Moreover, the quotient $\mathrm{H}_{\mathrm{sc}}(\mathtt{C}) = \mathrm{H}_{\mathrm{reg}}(\mathtt{C})/(\mathbf{L}-1)\mathrm{H}_{\mathrm{reg}}(\mathtt{C})$ is a commutative $\mathrm{K}(\mathrm{Var}/\mathbf{C})$-algebra. Thus the Poisson bracket of (2.3.14) preserves $\mathrm{H}_{\mathrm{reg}}(\mathtt{C})$, and descends to $\mathrm{H}_{\mathrm{sc}}(\mathtt{C})$.

We call $\mathrm{H}_{\mathrm{sc}}(\mathtt{C})$ the *semi-classical* Hall algebra of $\mathtt{C}$.

Proof. The proof given in [Bri12, Thm. 5.1] works for any heart $\mathtt{C} \subset \mathrm{D}(\mathcal{X})$. □

Next, we describe the codomain of the integration morphism. Fix a sign $\sigma \in \{\pm 1\}$, where the choice $+1$ corresponds to the topological Euler characteristic e and -1 corresponds to the Behrend-weighted Euler characteristic e_{B}.

Definition 2.3.25. The *quantum torus* is a commutative Poisson algebra $\mathbf{Q}_\sigma[\mathrm{N}(\mathcal{X})]$ generated as a vector space by symbols $\{t^\alpha \mid \alpha \in \mathrm{N}(\mathcal{X})\}$. Its product and Poisson bracket are defined on a basis by

$$t^\alpha \cdot t^\beta = \sigma^{\chi(\alpha,\beta)} t^{\alpha+\beta} \quad \text{and} \quad \{t^\alpha, t^\beta\} = \chi(\alpha,\beta) t^\alpha \cdot t^\beta \tag{2.3.15}$$

respectively, and extended $\mathbf{Q}$-linearly, where χ denotes the Euler form on $\mathrm{N}(\mathcal{X})$; see 2.1.6. The structures are well-defined because the Euler form is bilinear in general, and additionally anti-symmetric on a CY3 category.

There are two integration morphisms, one for each choice of sign $\sigma \in \{\pm\}$.

Definition 2.3.26. Define two $\mathbf{Q}$-linear maps $\mathrm{I}_\sigma \colon \mathrm{H}_{\mathrm{sc}}(\mathsf{C}) \to \mathbf{Q}_\sigma[\mathrm{N}(\mathcal{X})]$ as follows. Let Y be a variety and suppose that $f \colon \mathrm{Y} \to \underline{\mathsf{C}}$ factors through $\underline{\mathsf{C}}_\alpha \subset \underline{\mathsf{C}}$.

1. For $\sigma = 1$, we define $\mathrm{I}_1[\mathrm{Y} \to \underline{\mathsf{C}}_\alpha \subset \underline{\mathsf{C}}] = e(\mathrm{Y}) t^\alpha \in \mathbf{Q}_1[\mathrm{N}(\mathcal{X})]$.
2. For $\sigma = -1$, we define $\mathrm{I}_{-1}[\mathrm{Y} \to \underline{\mathsf{C}}_\alpha \subset \underline{\mathsf{C}}] = e_{\mathrm{B}}(\mathrm{Y} \to \underline{\mathsf{C}}_\alpha) t^\alpha \in \mathbf{Q}_{-1}[\mathrm{N}(\mathcal{X})]$. Here

$$e_{\mathrm{B}}\left(\mathrm{Y} \xrightarrow{s} \underline{\mathsf{C}}\right) := e(\mathrm{Y}, s^* \nu_{\mathfrak{M}}) = \sum_{k \in \mathbf{Z}} k e\left((\nu_{\mathfrak{M}} \circ s)^{-1}(k)\right) \tag{2.3.16}$$

where $\nu_{\mathfrak{M}} \colon \mathfrak{Num}_{\mathcal{X}} \to \mathbf{Z}$ denotes the Behrend function[5] of Corollary 2.2.13.

Theorem 2.3.27. The integration map I_σ is a map of Poisson algebras.

Proof. Bridgeland proves this for the case of $\sigma = 1$, and under a certain assumption for the case of $\sigma = -1$ as well [Bri12, Thm. 5.2]. As shown by Toda in [Tod16a, Thm. 2.8], this assumption holds true in our setting. The result follows. □

2.3.4 The graded Hall algebra

Many stacks or schemes that naturally occur in the theory of curve counting are not of finite type. For example, the Hilbert scheme of curves on $\mathcal{X}$, $\mathrm{Hilb}_{\leq 1}(\mathcal{X})$, is merely locally of finite type. However, the subscheme $\mathrm{Hilb}_\alpha(\mathcal{X})$ parametrising curves of a fixed numerical class $\alpha \in \mathrm{N}(\mathcal{X})$ is of finite type.

We extend the definition of the Hall algebra to include such objects.

Definition 2.3.28. The *graded Hall pre-algebra* $\mathrm{H}_{\mathrm{gr}}(\mathsf{C})$ is the $\mathbf{Q}$-vector space generated by symbols $[\mathrm{X} \to \underline{\mathsf{C}}]$, where X is an algebraic stack *locally* of finite type over $\mathbf{C}$ with affine geometric stabilisers, such that the restriction of X to $\underline{\mathsf{C}}_\alpha$ is of finite type for each $\alpha \in \mathrm{N}(\mathcal{X})$. We impose the same relations as before.

[5]Here we again anticipate the existence of the stack $\mathfrak{Num}_{\mathcal{X}}$, which we deduce by exhibiting an isomorphism $\mathfrak{Num}_{\mathrm{Y}} \to \mathfrak{Num}_{\mathcal{X}}$ in Proposition 4.2.2; in particular, their Behrend functions agree.

Remark 2.3.29. As the name suggests, the graded Hall pre-algebra is not quite an algebra, for essentially the same reason that the set of all formal expressions $\{\sum_{n\in\mathbf{Z}} aq^n\}$ is not a ring. Indeed, the product of two elements in $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$ may not lie in $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$, since the product may not be of finite type over all $\underline{\mathtt{C}}_\alpha$.

For convenience, let us make another definition.

Definition 2.3.30. We say that a full subcategory of $\mathtt{C}$ *defines an element* in the graded Hall algebra $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$ if its stack of objects is an open substack of $\underline{\mathtt{C}}$ (hence algebraic locally of finite type), which moreover is finite type when restricted to $\underline{\mathtt{C}}_\alpha$ for any $\alpha \in \mathrm{N}(\mathcal{X})$.

It is easy to give a sufficient condition for the product of two elements in $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$ to be well-defined and, hence, to define an element in $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$.

Lemma 2.3.31. Let $\mathrm{C} = \sum_\alpha \mathrm{C}_\alpha$, $\mathrm{D} = \sum_\alpha \mathrm{D}_\alpha$ be two elements of $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$ with C_α and D_α homogeneous of degree α. If the set

$$\{\alpha_1 + \alpha_2 = \alpha \mid \mathrm{C}_{\alpha_1} \neq 0 \neq \mathrm{D}_{\alpha_2}\}$$

is finite for every $\alpha \in \mathrm{N}(\mathcal{X})$, then the product of C and D exists in $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$.

Proof. A finite product of finite type stacks is again of finite type. □

One similarly defines graded versions of the regular subalgebra $\mathrm{H}_{\mathrm{gr,reg}}(\mathtt{C}) \subset \mathrm{H}_{\mathrm{gr}}(\mathtt{C})$, and the semi-classical quotient $\mathrm{H}_{\mathrm{gr,sc}}(\mathtt{C}) := \mathrm{H}_{\mathrm{gr,reg}}(\mathtt{C})/(\mathbf{L}-1)\mathrm{H}_{\mathrm{gr,reg}}(\mathtt{C})$. The latter comes equipped with a partially defined Poisson bracket and commutative product, and an integration map

$$\mathrm{I}_{\sigma,\mathrm{gr}}\colon \mathrm{H}_{\mathrm{gr,sc}}(\mathtt{C}) \to \mathbf{Q}_\sigma\{\mathrm{N}(\mathcal{X})\}, \tag{2.3.17}$$

where $\mathbf{Q}_\sigma\{\mathrm{N}(\mathcal{X})\}$ is the group of all formal expressions $\sum_{\alpha\in\mathrm{N}(\mathcal{X})} c_\alpha t^\alpha$ with coefficients $c_\alpha \in \mathbf{Q}$. We equip it with the same product and Poisson bracket as $\mathbf{Q}_\sigma[\mathrm{N}(\mathcal{X})]$ given in equation (2.3.15) on generators

$$t^\alpha \cdot t^\beta = \sigma^{\chi(\alpha,\beta)} t^{\alpha+\beta} \quad \text{and} \quad \{t^\alpha, t^\beta\} = \chi(\alpha,\beta) t^\alpha \cdot t^\beta \tag{2.3.18}$$

and extended $\mathbf{Q}$-linearly; again, note that these are only partially defined.

Proposition 2.3.32. The graded integration morphism $\mathrm{I}_{\sigma,\mathrm{gr}}$ is a morphism of Poisson algebras between any two elements for which the product or Poisson bracket are defined.

Proof. This follows by Theorem 2.3.27 by noting that if the condition of Lemma 2.3.31 holds for elements $\mathrm{C}, \mathrm{D} \in \mathrm{H}_{\mathrm{gr,sc}}$, then the same holds for their images in $\mathbf{Q}_\sigma\{\mathrm{N}(\mathcal{X})\}$. □

Remark 2.3.33. Whenever we write I_{gr} or $\mathbf{Q}[\mathrm{N}(\mathcal{X})] \subset \mathbf{Q}\{\mathrm{N}(\mathcal{X})\}$ we mean the version with $\sigma = -1$, i.e., the version leading to Behrend-weighted invariants and, hence, DT and PT invariants.

Remark 2.3.34. Working in $\mathrm{H}_{\mathrm{gr,sc}}(\mathtt{C})$ avoids having to choose various completions of $\mathrm{H}(\mathtt{C})$ to make sure elements are well defined, as is done in [Bri11, BG09b], and having to keep track of these as we pass from the one completion to the other. The cost of this choice is that at each step, we have to make sure that the product of objects under consideration is well defined.

For our applications, this is relatively easy to verify. In fact, we only wall-cross rank zero objects past rank -1 objects. Thus we only make use of the $\mathrm{H}_{\mathrm{rk}=0}(\mathtt{C})$-bimodule structure on $\mathrm{H}_{\mathrm{rk}=-1}(\mathtt{C})$ given by left and right multiplication.

2.4 The McKay correspondence

We discuss the derived McKay correspondence, both the original local statement of [BKR01] and the glued global statement of [CT08]. It is a derived equivalence

$$\Phi\colon \mathrm{D}(\mathrm{Y}) \to \mathrm{D}(\mathcal{X}) \tag{2.4.1}$$

where $\pi\colon \mathcal{X} \to \mathrm{X}$ is a three-dimensional Calabi–Yau orbifold as in Definition 1.2.13, with (quasi-)projective coarse moduli space X, and $f\colon \mathrm{Y} \to \mathrm{X}$ is a certain natural crepant resolution of singularities. Crucially, the equivalence preserves $\Phi(\mathcal{O}_{\mathrm{Y}}) = \mathcal{O}_{\mathcal{X}}$. Later, we establish a relation between the Hilbert schemes of Y and $\mathcal{X}$ via Φ.

2.4.1 Local McKay

Classically, the McKay correspondence relates the representation theory of a finite subgroup $\mathrm{G} \subset \mathrm{SL}_2(\mathbf{C})$ to the cohomology of the minimal resolution of the Kleinian singularity $\mathbf{C}^2/\mathrm{G}$. Here G acts by linear automorphisms on $\mathbf{C}^2$, and the only singularity is the origin. Originally, J. McKay observed the one-to-one correspondence between non-trivial irreducible representations of G and exceptional prime divisors of the minimal resolution Y of $\mathbf{C}^2/\mathrm{G}$ in [McK80]. The above is a geometric explanation of this observation, due to G. Gonzalez-Springberg and J.-L. Verdier [GSV83].

Example 2.4.1. Consider the A_n-singularity, where the group G acting on $\mathbf{C}^2$ is the cyclic group of order $n+1 \geq 2$ generated by the transformation

$$(x, y) \mapsto (\epsilon x, \epsilon^n y) \tag{2.4.2}$$

and where ϵ is a primitive $(n+1)^{\mathrm{st}}$ root of unity. The quotient variety

$$\mathrm{X} = \mathbf{C}^2/\mathrm{G} := \operatorname{Spec} \mathbf{C}[x,y]^{\mathrm{G}} \tag{2.4.3}$$

has an isolated singularity at the origin. It can be embedded as a hypersurface $\mathrm{X} \subset \mathbf{C}^3$ cut out by the equation $x^2 + y^2 + z^{n+1} = 0$ with isolated singularity at the origin. By repeatedly blowing up at the singular point, we obtain its minimal resolution $f\colon \mathrm{Y} \to \mathrm{X}$, whose exceptional locus is a chain of n rational (-2)-curves C_i. The corresponding resolution graph, obtained by taking a vertex for each C_i and an edge whenever $\mathrm{C}_i \cdot \mathrm{C}_j = 1$, is the Dynkin diagram of type A_n

The resolution is minimal in that all other resolutions of singularities factor through this one or, equivalently by Castelnuovo's theorem, that there are no rational (-1)-curves on the resolution that can be blown down to a smooth point.

Minimal resolutions of three-dimensional varieties do not exist in general. Crepant resolutions, a natural generalisation, always exist provided the singularities are mild.

Definition 2.4.2. A variety is *Gorenstein* if it is Cohen–Macaulay and its canonical sheaf is a line bundle.

Remark 2.4.3. If $\mathrm{G} \subset \mathrm{SL}_n(\mathbf{C})$ is a finite subgroup then the quotient variety $\mathbf{C}^n/\mathrm{G}$ is Gorenstein by K. Watanabe [Wat74]. In particular, Kleinian singularities are Gorenstein.

Definition 2.4.4. A resolution of singularities $f\colon \mathrm{Y} \to \mathrm{X}$ of a normal variety X with Gorenstein singularities is called *crepant* if $f^*\omega_{\mathrm{X}} = \omega_{\mathrm{Y}}$.

Example 2.4.5. The minimal resolutions of a Kleinian surface singularity is crepant.

In the celebrated [BKR01], the McKay correspondence is lifted to an equivalence of derived categories whilst simultaneously establishing the existence of crepant resolutions of three-dimensional varieties with Gorenstein singularities.[6]

Although the result is proven in a greater generality, we restrict to the following statement. Let $\mathrm{G} \subset \mathrm{SL}_3(\mathbf{C})$ be a finite subgroup, so the quotient $\mathrm{X} = \mathbf{C}^3/\mathrm{G}$ has only Gorenstein singularities. A candidate crepant resolution was introduced by I. Nakamura in [Nak01].

Definition 2.4.6. The G-Hilbert scheme G-Hilb$(\mathbf{C}^3)$ parametrises G-clusters on $\mathbf{C}^3$. Such a cluster is a G-invariant zero-dimensional subscheme $\mathrm{Z} \subset \mathbf{C}^3$ such that its space of global sections $\mathrm{H}^0(\mathcal{O}_{\mathrm{Z}})$, which carries an induced G-action, is isomorphic to the regular representation $\mathbf{C}[\mathrm{G}]$ of G as G-representations.

[6]This fact had earlier been verified via a case-by-case analysis; see [Roa96] and references therein.

Remark 2.4.7. A G-cluster has length $|G|$. Any free G-orbit yields a G-cluster.

Furthermore, there is a Hilbert–Chow morphism

$$f\colon Y = G-\mathrm{Hilb}(\mathbf{C}^3) \to \mathbf{C}^3/G = X \tag{2.4.4}$$

that on closed points sends a G-cluster to the orbit supporting it. It is a proper birational morphism. The G-Hilbert scheme is the analogue of the minimal resolution of $\mathbf{C}^2/G$. It is a fine moduli space in that there exists a universal G-cluster $\mathcal{Z} \subset \mathbf{C}^3 \times Y$.

Example 2.4.8. Consider the setting of Example 2.4.1 for $n = 1$. Let $p \in \mathbf{C}^2$ denote a point and let $Z \subset \mathbf{C}^2$ be a $\mathbf{Z}_2$-cluster such that Z is sent to the orbit of p by f. There are two types of $\mathbf{Z}_2$-cluster.

1. If $p = 0$, then Z is supported set-theoretically on the orbit $\{0\}$. Any $\mathbf{Z}_2$-invariant subscheme of $\mathbf{C}^2$ supported at the origin is determined by a $\mathbf{Z}_2$-invariant ideal $I \leq \mathbf{C}[x,y]$ such that

$$q\colon \mathbf{C}[x,y] \twoheadrightarrow \mathbf{C}[x,y]/(x^2,xy,y^2) \twoheadrightarrow \mathbf{C}[x,y]/I \tag{2.4.5}$$

and $\mathbf{C}[x,y]/I = \mathbf{C}[\mathbf{Z}_2]$ as $\mathbf{Z}_2$-representations. Thus $I_{(a:b)} = (x^2, xy, y^2, ax+by)$ where $(a:b) \in \mathbf{P}^1$, corresponding to a choice of tangent vector to $\mathbf{C}^2$ at the origin, up to scaling. Note that $\mathbf{C}[x,y]/I_{(a:b)} = \mathbf{C}\cdot 1 \oplus \mathbf{C}\cdot \rho_-$ where ρ_- is the non-trivial irreducible representation of $\mathbf{Z}_2$.

2. If $p \neq 0$, the orbit $\mathbf{Z}_2 \cdot p$ of p is free and the only $\mathbf{Z}_2$-cluster is $Z = \mathbf{Z}_2 \cdot p$.

Extending this argument to families yields an isomorphism $\mathbf{Z}_2\text{-Hilb}(\mathbf{C}^2) \cong \mathrm{Tot}_{\mathbf{P}^1}(\mathcal{O}(-2))$ with the total space of the line bundle $\mathcal{O}(-2)$ on $\mathbf{P}^1$. Alternatively, this variety can be realised by blowing up $\mathbf{C}^2/\mathbf{Z}_2$ at the origin.

The global quotient stack $[\mathbf{C}^3/G]$ is also a tautological resolution of its coarse moduli space C^3/G. It is a smooth Deligne–Mumford stack since the morphism $a\colon \mathbf{C}^3 \twoheadrightarrow [\mathbf{C}^3/G]$ is an étale surjection. Moreover, there is an equivalence of categories

$$\mathrm{Coh}([\mathbf{C}^3/G]) \cong \mathrm{Coh}^G(\mathbf{C}^3) \tag{2.4.6}$$

between coherent sheaves on the stack and G-equivariant coherent sheaves on $\mathbf{C}^3$. The category $\mathrm{Coh}^G(\mathbf{C}^3)$ is the analogue of the representation theory of G on $\mathbf{C}^2$.

The derived McKay correspondence [BKR01, Thm. 1.2] is the following

Theorem 2.4.9 ([BKR01])**.** Let $n = 2, 3$, let $G \subset \mathrm{SL}_n(\mathbf{C})$ be a finite subgroup with trivial irreducible representation ρ_0, and let $X = \mathbf{C}^n/G$ be the quotient. Let Y be the irreducible component of G-Hilb$(\mathbf{C}^n)$ containing the free G-orbits, and let $\mathcal{Z} \subset \mathrm{Y} \times \mathbf{C}^n$ denote the universal G-cluster. We write p and q for the natural projections from $\mathcal{Z}$ to $\mathbf{C}^n$ and Y respectively.

Then Y = G-Hilb$(\mathbf{C}^n)$ is irreducible, the Hilbert–Chow morphism $f\colon \mathrm{Y} \to \mathrm{X}$ is a crepant resolution, in particular Y is smooth, and the Fourier–Mukai functor

$$\mathbf{R}q_* \circ p^*\colon \mathrm{D}(\mathrm{Y}) \longrightarrow \mathrm{D}^{\mathrm{G}}(\mathbf{C}^n) = \mathrm{D}([\mathbf{C}^n/\mathrm{G}]), \tag{2.4.7}$$

is an equivalence of triangulated categories that sends $\mathcal{O}_{\mathrm{Y}}$ to $\mathcal{O}_{\mathbf{C}^3} \otimes \rho_0$.

Example 2.4.10. Consider the previous example. Denote the resolution by $\mathrm{Y} = \mathbf{Z}_2$-Hilb$(\mathbf{C}^2) = \mathrm{Tot}_{\mathbf{P}^1}(\mathcal{O}(-2))$ which is irreducible. Let $p \in \mathrm{Y}$ be a closed point The equivalence reads $\Phi\colon \mathrm{D}(\mathrm{Y}) \to \mathrm{D}^{\mathbf{Z}_2}(\mathbf{C}^2)$. Hence $\Phi(\mathcal{O}_p) = \mathcal{O}_{\mathcal{Z}_p}$ where $\mathcal{Z}_p$ denotes the restriction of $\mathcal{Z}$ to the $\mathbf{Z}_2$-cluster on $\mathbf{C}^2$ corresponding to $p \in \mathrm{Y}$.

Let ρ^+ and ρ^- denote the trivial and non-trivial irreducible representations of $\mathbf{Z}_2$. If $p \in \mathrm{Y}$ is a $\mathbf{Z}_2$-cluster supported at the origin of $\mathbf{C}^2$, there is an exact sequence

$$0 \to \mathcal{O}_0^- \to \mathcal{O}_{\mathcal{Z}_p} \to \mathcal{O}_0^+ \to 0 \tag{2.4.8}$$

of $\mathbf{Z}_2$-equivariant sheaves on $\mathbf{C}^2$ where $\mathcal{O}_0^\pm = \mathcal{O}_0 \otimes \rho^\pm$. To figure out where the McKay equivalence sends $\mathcal{O}_0^\pm$, note that it preserves the Euler characteristic. We have

$$\chi(\mathcal{O}_{\mathbf{C}^3} \otimes \rho^+, \mathcal{O}_0^-) = 0 \quad \text{and} \quad \chi(\mathcal{O}_{\mathbf{C}^3} \otimes \rho^+, \mathcal{O}_0^+) = 1. \tag{2.4.9}$$

Moreover, p corresponds to a point on the zero section $\mathbf{P}^1$ of $\mathrm{Tot}_{\mathbf{P}^1}(\mathcal{O}(-2))$. Consider the exact triangle $\mathcal{O}_{\mathbf{P}^1}(-1) \to \mathcal{O}_p \to \mathcal{O}_{\mathbf{P}^1}(-2)[1]$ in $\mathrm{D}(\mathrm{Y})$. Comparing to equation (2.4.8), we deduce that

$$\Phi(\mathcal{O}_{\mathbf{P}^1}(-1)) = \mathcal{O}_0^- \quad \text{and} \quad \Phi(\mathcal{O}_{\mathbf{P}^1}(-2)[1]) = \mathcal{O}_0^+. \tag{2.4.10}$$

In particular, Φ does *not* restrict to an equivalence $\mathrm{Coh}(\mathrm{Y}) \cong \mathrm{Coh}(\mathcal{X})$.

2.4.2 Global McKay

We results of the local McKay correspondence are globalised in [CT08], as we now describe. Let $\mathcal{X}$ be a smooth three-dimensional Deligne–Mumford stack that has generically trivial stabilisers, and assume that its coarse moduli space $\pi\colon \mathcal{X} \to \mathrm{X}$ is (quasi)projective. In addition, assume that $\omega_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}}$ and $\mathrm{H}^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = 0$. We call such an object a CY3 orbifold with (quasi)projective coarse moduli space.

A candidate for a crepant resolution of X is given by a certain irreducible component of Hilb$(\mathcal{X})$, which is the fine moduli space representing the Quot functor in the sense of [OS03] parametrising closed points on $\mathcal{X}$. Let $\mathrm{Y} \subset \mathrm{Hilb}(\mathcal{X})$ denote the irreducible component containing the non-stacky points.[7] The morphism $\pi\colon \mathcal{X} \to \mathrm{X}$ induces a morphism $\mathrm{Hilb}(\mathcal{X}) \to \mathrm{Hilb}(\mathrm{X})$ and by restriction a morphism $f\colon \mathrm{Y} \to \mathrm{X}$.

$$\begin{tikzcd} \mathcal{X} \arrow[dr, "\pi"'] & & \mathrm{Y} \arrow[dl, "f"] \\ & \mathrm{X} & \end{tikzcd} \tag{2.4.11}$$

The condition that $\mathcal{X}$ has generically trivial stabilisers implies that f is birational. Since Y represents a moduli functor it comes equipped with a universal quotient $\mathcal{O}_{\mathcal{X}\times\mathrm{Y}} \twoheadrightarrow \mathcal{O}_{\mathcal{Z}}$.

The global version of the derived McKay correspondence is the following

Theorem 2.4.11 ([CT08])**.** Let $\pi\colon \mathcal{X} \to \mathrm{X}$ and $f\colon \mathrm{Y} \to \mathrm{X}$ be as above. We write p and q for the natural projections from $\mathcal{Z}$ to $\mathcal{X}$ and Y respectively. Then Y is smooth and the morphism $f\colon \mathrm{Y} \to \mathrm{X}$ is a crepant resolution. Moreover, the Fourier–Mukai functor

$$\Phi := \mathbf{R}q_* \circ p^*\colon \mathrm{D}(\mathrm{Y}) \longrightarrow \mathrm{D}(\mathcal{X}) \tag{2.4.12}$$

is an equivalence of triangulated categories.

An example of this, a family version of Example 2.4.1, is discussed in Chapter 3.

Remark 2.4.12. The condition that $\mathcal{X}$ has generically trivial stabilisers means that $\mathcal{X}$ is a quotient stack étale locally on X. To be precise, it means any point $x \in \mathrm{X}$ admits an étale neighbourhood $e\colon \mathbf{C}^3 \to \mathrm{X}$ such that $e(0) = x$ and

$$\mathcal{X} \times_{\mathrm{X}} \mathbf{C}^3 = [\mathbf{C}^3/\mathrm{G}].$$

Here G is the finite stabiliser group of the point x. Since $\mathcal{X}$ is Calabi–Yau, we have $\mathrm{G} \subset \mathrm{SL}_3(\mathbf{C})$. The resolution pulls back to $f\colon \mathrm{G\text{-}Hilb}(\mathbf{C}^3) \to \mathbf{C}^3/\mathrm{G}$ by [CT08, Lem. 2.2].

Thus, pulling back the diagram in equation (2.4.11) to such an étale neighbourhood reduces it to the setting of the local McKay correspondence. The results of the global theorem are then obtained from the local one. For example, checking that a particular Fourier–Mukai kernel defines an equivalence may be done locally.

The crepant resolution conjecture of [BCY12] conjectures a comparison between the counts of curves on $\mathcal{X}$ and Y in a more restricted setting.

Definition 2.4.13. Let $\mathcal{X}$ be a CY3 orbifold with (quasi)projective coarse moduli space X, and let $f\colon \mathrm{Y} \to \mathrm{X}$ denote the natural crepant resolution. We say that $\mathcal{X}$ satisfies the *hard Lefschetz condition* if $\dim f^{-1}(x) \le 1$ for all $x \in \mathrm{X}$.

[7] In the local case, this corresponds to the component of $\mathrm{G\text{-}Hilb}(\mathbf{C}^3)$ containing the free G-orbits.

Originally, an equivalent condition on the orbifold was introduced by J. Fernandez in [Fer06] that describes when the hard Lefschetz Theorem holds for orbifold cohomology.[8] This explains the name of the condition.

There is the following characterisation of this condition given in [BG09a, Lem. 24].

Lemma 2.4.14. Let $\mathrm{G} \subset \mathrm{SU}(3)$ be a finite subgroup. The following are equivalent.

1. The orbifold $[\mathbf{C}^3/\mathrm{G}]$ satisfies the hard Lefschetz condition [Fer06, Def. 1.1].

2. G is a finite subgroup of $\mathrm{SO}(3) \subset \mathrm{SU}(3)$ or $\mathrm{SU}(2) \subset \mathrm{SU}(3)$.

Remark 2.4.15. In terms of the resolution of singularities, the hard Lefschetz condition means that f may contract a divisor to a curve, but not to a point. Thus two types of curves in the exceptional locus can exist: those that map to curves in the singular locus of X, and those that get contracted to a point.

Remark 2.4.16. An example of a geometry violating the hard Lefschetz condition is given by the quotient singularity of type $\frac{1}{3}(1,1,1)$ at the origin of $\mathbf{C}^3$. The corresponding crepant resolution is given by

$$\mathbf{Z}_3\text{-Hilb}(\mathbf{C}^3) \cong \mathrm{Tot}(\omega_{\mathbf{P}^2}) \longrightarrow \mathbf{P}^2, \tag{2.4.13}$$

the total space of the canonical bundle of the exceptional locus $\mathrm{E} = \mathbf{P}^2$. In [Tod16b], Y. Toda has worked out the Pandharipande-Thomas invariants in this situation, which naturally involves *surface* classes supported on the exceptional locus $\mathbf{P}^2$. His results crucially depend on work of A. Bayer and E. Macrì [BM11] describing the Bridgeland stability manifold of local $\mathbf{P}^2$.

Thus we additionally assume that $\mathcal{X}$ satisfies the hard Lefschetz condition. Then, f has relative dimension at most one, X is a (quasi)projective Gorenstein Calabi–Yau threefold with rational quotient singularities $\mathbf{R}f_*\mathcal{O}_\mathrm{Y} = \mathcal{O}_\mathrm{X}$, and the singular locus of X is one-dimensional. Finally, Y is a smooth Calabi–Yau threefold by the crepancy of f.

We are now in a position to explain the claims made in Section 1.3 concerning the *exceptional* and *multi-regular* numerical classes in the numerical Grothendieck groups of Y and $\mathcal{X}$ respectively. Before we do so we recall diagram 1.3.4, summarising the numerical setup, for the convenience of the reader:

$$\begin{tikzcd} \mathrm{N}_0(\mathrm{Y}) \arrow[r, hook] & \mathrm{N}_{\mathrm{exc}}(\mathrm{Y}) \arrow[r, hook] \arrow[d, equal, "\phi"] & \mathrm{N}_{\leq 1}(\mathrm{Y}) \arrow[d, equal, "\phi"] & \\ & \mathrm{N}_0(\mathcal{X}) \arrow[r, hook] & \mathrm{N}_{\mathrm{mr}}(\mathcal{X}) \arrow[r, hook] & \mathrm{N}_{\leq 1}(\mathcal{X}) \end{tikzcd} \tag{2.4.14}$$

[8]Orbifold cohomology is the ordinary cohomology of the inertia stack of $\mathcal{X}$ with a shifted grading.

Recall that we define the *exceptional* classes on Y via $\mathrm{N}_{\mathrm{exc}}(\mathrm{Y}) := \phi^{-1}(\mathrm{N}_0(\mathcal{X}))$ and the *multi-regular* classes via $\mathrm{N}_{\mathrm{mr}}(\mathcal{X}) := \phi(\mathrm{N}_{\leq 1}(\mathrm{Y}))$. The following result gives an interpretation of these classes, and explains the etymology of *multi-regular*.

Lemma 2.4.17. Let $\mathcal{X}$ be a CY3 orbifold satisfying the hard Lefschetz condition with projective coarse moduli space $\pi\colon \mathcal{X} \to \mathrm{X}$, and let $f\colon \mathrm{Y} \to \mathrm{X}$ be the natural crepant resolution given by the McKay correspondence. Then

1. the group $\mathrm{N}_{\mathrm{exc}}(\mathrm{Y})$ is generated by the classes of sheaves supported on the one-dimensional exceptional fibres of f, and
2. the group $\mathrm{N}_{\mathrm{mr}}(\mathcal{X})$ is generated by classes of sheaves supported in dimension at most one where at a general point p of each curve in the support, the associated representation of the stabiliser group $\mathrm{G}_p \subset \mathrm{SL}_3(\mathbf{C})$ of p on $\mathbf{C}^3$ (étale-locally in the sense of Remark 2.4.12) is a multiple of the regular representation.

In order to prove this result, we use the following lemma.

Lemma 2.4.18. The following two properties hold.

1. If $\mathcal{X}$ is a CY3 orbifold that satisfies the hard Lefschetz condition, then its stacky locus is one-dimensional.
2. Let $\mathrm{G} \subset \mathrm{SL}_2(\mathbf{C})$ be a finite subgroup, let ρ_{reg} denotes its regular representation, and let $0 \neq p \in \mathbf{C}^2$ be a point. We have $[\mathcal{O}_p] = [\mathcal{O}_0 \otimes \rho_{\mathrm{reg}}]$ in the compactly supported numerical Grothendieck-group $\mathrm{N}_c([\mathbf{C}^2/\mathrm{G}])$ of $[\mathbf{C}^2/\mathrm{G}]$.

Proof. We treat the two claims separately.

The first claim may be checked after étale base change, so we argue as follows. Let $p \in \mathcal{X}$ be a stacky point, and let

$$\begin{tikzcd}[\mathbf{C}^3/\mathrm{G}_p] \arrow[dr, "\pi"'] & & \mathrm{G}_p\text{-Hilb}(\mathbf{C}^3) \arrow[dl, "f"] \\ & \mathbf{C}^3/\mathrm{G}_p & \end{tikzcd} \tag{2.4.15}$$

be the pullback of the bottom part of diagram 2.4.18 to an étale neighbourhood of $\pi(p) \in \mathrm{X}$ as in Remark 2.4.12. Here G_p denotes the stabiliser group of $p \in \mathcal{X}$. Take an element $g \in \mathrm{G}_p \subset \mathrm{SL}_3(\mathbf{C})$ that is not the identity. Since $\det(g) = 1$, it has either zero or one eigenvalue equal to 1. By the hard Lefschetz condition $\mathrm{G}_p \subset \mathrm{SL}_2(\mathbf{C}) \times \mathbf{C}$ or $\mathrm{G}_p \subset \mathrm{SO}(3)$, so it has to have precisely one such eigenvalue. Thus the fixed point locus of $g\colon \mathbf{C}^3 \to \mathbf{C}^3$, and hence the singular locus of $\mathbf{C}^3/\mathrm{G}_p$ and the stacky locus of $[\mathbf{C}^3/\mathrm{G}_p]$, is one-dimensional.

As for the second claim, note that the pushforward along $[\mathbf{C}^2/\mathrm{G}] \to [\mathrm{pt}/\mathrm{G}]$ takes G-equivariant sections $\mathrm{H}^0_{\mathrm{G}}(\mathbf{C}^2, -)$. It induces an isomorphism

$$f\colon \mathrm{N}_c\left([\mathbf{C}^2/\mathrm{G}]\right) \xrightarrow{\sim} \mathrm{N}_c\left([\mathrm{pt}/\mathrm{G}]\right) \tag{2.4.16}$$

because $\mathbf{C}^2$ is affine. Note that $[\mathcal{O}_0 \otimes \rho_{\mathrm{reg}}]$ is sent to the regular G-representation ρ_{reg} by f, since a coherent sheaf on $[\mathrm{pt}/\mathrm{G}]$ corresponds to a finite dimensional G-representation; see equation (2.4.6). Thus it suffices to show that $\mathrm{H}^0_{\mathrm{G}}(\mathbf{C}^2, \mathcal{O}_p)$ is isomorphic to ρ_{reg} as G-representations for all $0 \neq p \in \mathbf{C}^2$, where we think of $\mathcal{O}_p$ as a G-sheaf on $\mathbf{C}^2$.

Since $0 \in \mathbf{C}^2$ is the only point with a non-trivial stabiliser, $\mathcal{O}_p$ decomposes as a G-equivariant sheaf into a direct sum of skyscraper sheaves

$$\mathcal{O}_p = \mathcal{O}_{p_1} \oplus \ldots \oplus \mathcal{O}_{p_{|\mathrm{G}|}} \tag{2.4.17}$$

where the points $p_1, \ldots, p_{|\mathrm{G}|}$ form a *free* G-orbit in $\mathbf{C}^2$. But then $\mathrm{H}^0_{\mathrm{G}}(\mathbf{C}^2, \mathcal{O}_p)$ has a basis consisting of $|\mathrm{G}|$ vectors that form a free G-orbit, and the claim follows. □

Proof of Lemma 2.4.17. We treat the two claims separately.

1. Note that the kernel $\mathcal{O}_{\mathcal{Z}}$ of the McKay equivalence $\Phi\colon \mathrm{D}(\mathrm{Y}) \to \mathrm{D}(\mathcal{X})$ is supported on the fibre product $\mathcal{X} \times_{\mathrm{X}} \mathrm{Y}$, where $\mathcal{O}_{\mathcal{X}\times\mathrm{Y}} \twoheadrightarrow \mathcal{O}_{\mathcal{Z}}$ is the universal quotient. The functor Φ is given by $\Phi(\mathrm{A}) = \mathbf{R}p_{\mathcal{X},*}\left(\mathcal{O}_{\mathcal{Z}} \otimes p_{\mathrm{Y}}^*\mathrm{A}\right)$ and induces $\phi\colon \mathrm{N}(\mathrm{Y}) \to \mathrm{N}(\mathcal{X})$.

$$\begin{tikzcd}[column sep=small] \mathcal{X} \times_{\mathrm{X}} \mathrm{Y} \arrow[r, hook, "i"] & \mathcal{X} \times \mathrm{Y} \arrow[dl, "p_{\mathcal{X}}"'] \arrow[dr, "p_{\mathrm{Y}}"] & \\ \mathcal{X} \arrow[dr, "\pi"'] & & \mathrm{Y} \arrow[dl, "f"] \\ & \mathrm{X} & \end{tikzcd} \tag{2.4.18}$$

   Let $\alpha = [\mathrm{F}]$ be the class of a sheaf supported on an exceptional fibre of f, say over the point $x \in \mathrm{X}$. Then $p_{\mathrm{Y}}^*(\mathrm{F})$ is supported on $\mathcal{X} \times f^{-1}(x)$ and $\mathcal{O}_{\mathcal{Z}} \otimes p_{\mathrm{Y}}^*(\mathrm{F})$ is supported on $\pi^{-1}(x) \times f^{-1}(x)$. Pushing forward along $p_{\mathcal{X}}$, we find that $\Phi(\mathrm{F})$ is supported on the zero-dimensional locus $\pi^{-1}(x)$, and so $\phi(\alpha) \in \mathrm{N}_0(\mathcal{X})$. A similar reasoning shows that $\phi^{-1}([\mathrm{Q}])$ for $\mathrm{Q} \in \mathrm{Coh}_0(\mathcal{X})$ is a formal difference of classes of sheaves supported on the exceptional fibres of f, as required.

2. Let $\alpha \in \phi(\mathrm{N}_{\leq 1}(\mathrm{Y}))$ and decompose it into a sum of classes of sheaves on $\mathcal{X}$. Let $\mathrm{C} \subset \mathcal{X}$ be an irreducible component of the stacky locus, which is one-dimensional by the first part of Lemma 2.4.18, and let α_{C} consist of those parts of α supported on C. If $\alpha_{\mathrm{C}} \in \mathrm{N}_0(\mathcal{X})$ there is nothing to show, so assume that α_{C} is one-dimensional.

Let $p \in C$ be a general point of C. Taking an étale-local transverse slice of C at p looks like

$$[\mathbf{C}^2/G_p] \longrightarrow \mathbf{C}^2/G_p \longleftarrow G_p\text{-Hilb}(\mathbf{C}^2) := Y_p. \tag{2.4.19}$$

By assumption, the McKay transform of the restriction of α_C to this slice lies in $N_0(Y_p)$. But $N_0(Y_p) \cong \mathbf{Z}\cdot[\mathcal{O}_y]$ where $y \in Y_p$ is a non-singular point, which corresponds to $\mathcal{O}_p \otimes \rho_{\mathrm{reg}}$ by the second part of Lemma 2.4.18.

Conversely, let $F \in \mathrm{Coh}_{\leq 1}(\mathcal{X})$ be a sheaf with the property of part 2. We claim that $\phi^{-1}([F]) \in N_{\leq 1}(Y)$. If F is zero-dimensional, this follows from part 1 since the exceptional fibres are one-dimensional. Now, fix a general point p of an irreducible curve C in the support of F. If C intersects the one-dimensional stacky locus of $\mathcal{X}$ in finitely many points, the claim again follows from part 1.

Finally, assume that C lies in the stacky locus of $\mathcal{X}$. By part 1, the behaviour at any single point in C yields an at most one-dimensional class in N(Y). Thus it suffices to treat the generic case. By assumption, étale-locally around a general point $p \in C$, the associated representation on $\mathbf{C}^3$ of the stabiliser group $G_p \subset SL_3(\mathbf{C})$ of p is multi-regular. This means that étale-locally around p, F looks like $\mathcal{O}_C \otimes V$ where V is a multiple of the regular representation of G_p. The claim then follows from the argument of the second part of Lemma 2.4.18.

This completes the proof. □

Finally, in [Cal16b], J. Calabrese identifies the image of $\mathrm{Coh}(\mathcal{X}) \subset D(\mathcal{X})$ under Φ. To state the result, we introduce two torsion pairs on the abelian category Coh(Y). The tilt at one of these hearts is identified with $\mathrm{Coh}(\mathcal{X})$. Consider the full subcategories

$$\begin{aligned} {}^{-1}\mathtt{T} &= \{T \in \mathrm{Coh}(Y) \mid \mathbf{R}^1 f_* T = 0,\ \mathrm{Hom}(T, \mathtt{A}_f) = 0\} \\ {}^{-1}\mathtt{F} &= \{F \in \mathrm{Coh}(Y) \mid f_*(F) = 0\}, \end{aligned} \tag{2.4.20}$$

and

$$\begin{aligned} {}^{0}\mathtt{T} &= \{T \in \mathrm{Coh}(Y) \mid \mathbf{R}^1 f_* T = 0\} \\ {}^{0}\mathtt{F} &= \{F \in \mathrm{Coh}(Y) \mid f_*(F) = 0,\ \mathrm{Hom}(\mathtt{A}_f, F) = 0\}, \end{aligned} \tag{2.4.21}$$

of Coh(Y), where $\mathtt{A}_f := \{E \in \mathrm{Coh}(Y) \mid \mathbf{R}f_*(E) = 0\}$ is an abelian subcategory [Bri02].

Definition 2.4.19. For $p = 0, 1$, we write ${}^p\,\mathrm{Per}(Y/X) := \langle {}^p\mathtt{F}[1], {}^p\mathtt{T}\rangle$ for the tilt at the torsion pair $({}^p\mathtt{T}, {}^p\mathtt{F})$. This is the category of *p-perverse coherent sheaves* introduced by T. Bridgeland in [Bri02]. We are mainly interested in $\mathrm{Per}(Y/X) := {}^0\,\mathrm{Per}(Y/X)$.

The following result is [Cal16b, Thm. 1.4].

Theorem 2.4.20. Let $\mathcal{X}$ be a CY3 orbifold with (quasi)projective coarse moduli space. Assume that $\mathcal{X}$ satisfies the hard Lefschetz condition. The McKay equivalence identifies the structure sheaves $\Phi(\mathcal{O}_{\mathrm{Y}}) = \mathcal{O}_{\mathcal{X}}$ and restricts to an equivalence of abelian categories

$$\Phi\colon \mathrm{Per}(\mathrm{Y}/\mathrm{X}) \to \mathrm{Coh}(\mathcal{X}). \tag{2.4.22}$$

If in addition X, and hence Y, is projective, then we have $\mathbf{R}f_* = \pi_* \circ \Phi$.

This description of $\Phi^{-1}(\mathrm{Coh}(\mathcal{X}))$ is key to identify the result of the wall-crossing computation in Chapter 5 that underlies the proof of the crepant resolution conjecture.

2.4.3 Bryan–Steinberg invariants

Let $\mathcal{X}$ be a smooth three-dimensional Calabi–Yau orbifold satisfying the hard Lefschetz condition, and assume that its coarse moduli space $\pi\colon \mathcal{X} \to \mathrm{X}$ is quasi-projective. Let $f\colon \mathrm{Y} \to \mathrm{X}$ denote the crepant resolution of the previous section, and let $\Phi\colon \mathrm{D}(\mathrm{Y}) \to \mathrm{D}(\mathcal{X})$ denote the McKay derived equivalence where $\Phi(\mathcal{O}_{\mathrm{Y}}) = \mathcal{O}_{\mathcal{X}}$.

In [BS16], J. Bryan and D. Steinberg define curve-counting invariants associated to the crepant resolution $f\colon \mathrm{Y} \to \mathrm{X}$ in the McKay setting. Here we collect a number of results about these invariants, and we compute some examples.

First, we recall the definition from 1.2.15.

Definition 2.4.21. An *f-stable pair* or *Bryan–Steinberg pair* (G, s) consists of a one-dimensional sheaf G on Y and a section $s \in \mathrm{H}^0(\mathrm{Y}, \mathrm{G})$. This data satisfies two stability requirements:

(i) $\mathrm{coker}(s)$ pushes down to a zero-dimensional sheaf, i.e., $\mathrm{coker}(s) \in \mathtt{T}_f$, and

(ii) G admits no maps from such sheaves, i.e., $\mathrm{Hom}(\mathtt{T}_f, \mathrm{G}) = 0$,

where $\mathtt{T}_f \coloneqq \{\mathrm{T} \in \mathrm{Coh}_{\leq 1}(\mathrm{Y}) \mid \mathbf{R}f_*(\mathrm{T}) \in \mathrm{Coh}_0(\mathrm{X})\}$. Two BS pairs (G, s) and (G', s') are *isomorphic* if there exists an isomorphism $\phi\colon \mathrm{G} \to \mathrm{G}'$ such that $\phi \circ s = s'$.

Example 2.4.22. Any stable pair $(s\colon \mathcal{O}_{\mathrm{Y}} \to \mathrm{G})$ is an f-stable pair, as long as G is not exclusively supported on exceptional curves; see [BS16, Prop. 18]. We describe a family of strict f-stable pairs in the following example.

Let T be the total space $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1$, and let $\mathbf{Z}_2$ act fibre-wise on T by sending fibre coordinates $(x, y) \mapsto (-x, -y)$. Let $\mathcal{X} = [\mathrm{T}/\mathbf{Z}_2]$ be the stacky quotient, let $\mathrm{X} = \mathrm{T}/\mathbf{Z}_2$ be the usual quotient, and let $f\colon \mathrm{Y} \to \mathrm{X}$ be the natural crepant resolution of singularities given by [BKR01]. This is a trivial $\mathbf{P}^1$-family of A_1-surface singularities. The crepant resolution Y of the coarse moduli space X is the fibre-wise resolution of these surface singularities. Hence Y may be identified with the total space of $\mathcal{O}(-2,-2) \to \mathbf{P}^1 \times \mathbf{P}^1$, and f contracts its zero section $\mathrm{pr}_1 \colon \mathrm{E} = \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ fibre-wise.

Let $\mathrm{V} \subset \mathrm{E}$ denote the exceptional fibre above a singular point $p \in \mathrm{X}$, and let $\mathrm{H} \subset \mathrm{E}$ denote a horizontal section. Note that $\mathcal{O}_{\mathrm{V}}(-1) \in \mathtt{T}_f$ but $\mathcal{O}_{\mathrm{V}}(-2) \notin \mathtt{T}_f$ because

$$\mathbf{R}f_*\mathcal{O}_{\mathrm{V}}(-2) = \big(\mathbf{R}^1 f_* \mathcal{O}_{\mathrm{V}}(-2)\big)[-1] = \mathcal{O}_p[-1] \notin \mathrm{Coh}_0(\mathcal{X}). \tag{2.4.23}$$

Let $\mathrm{Z} \subset \mathrm{E}$ be the closed subscheme given by the union of V and H such that we have a short exact sequence $0 \to \mathcal{O}_{\mathrm{E}}(-1,-1) \to \mathcal{O}_{\mathrm{E}} \to \mathcal{O}_{\mathrm{Z}} \to 0$. Consider $\mathrm{G} \coloneqq \mathcal{O}_{\mathrm{Z}} \otimes \mathcal{O}_{\mathrm{E}}(-1,1)$. Since Z contains both V and H as closed subschemes, we obtain two exact sequences

$$\begin{gathered} 0 \to \mathcal{O}_{\mathrm{V}}(-2) \to \mathrm{G} \to \mathcal{O}_{\mathrm{H}}(1) \to 0 \\ 0 \to \mathcal{O}_{\mathrm{H}} \xrightarrow{i} \mathrm{G} \to \mathcal{O}_{\mathrm{V}}(-1) \to 0. \end{gathered} \tag{2.4.24}$$

Write $p \colon \mathcal{O}_{\mathrm{Y}} \twoheadrightarrow \mathcal{O}_{\mathrm{Z}}$. We claim that $s = i \circ p \colon \mathcal{O}_{\mathrm{Y}} \twoheadrightarrow \mathcal{O}_{\mathrm{H}} \hookrightarrow \mathrm{G}$ is a strict f-stable pair. Indeed, by the second exact sequence we have $\mathrm{coker}(s) = \mathrm{coker}(i) = \mathcal{O}_{\mathrm{V}}(-1) \in \mathtt{T}_f$ as required. By the first exact sequence we deduce $\mathrm{Hom}(\mathtt{T}_f, \mathrm{G}) = 0$, so the claim follows.

Take $n \geq 1$. Tensoring i by the line bundle $\mathcal{O}_{\mathrm{E}}(0,n)$ induces strict f-stable pairs $s_{n+1} \colon \mathcal{O}_{\mathrm{Y}} \twoheadrightarrow \mathcal{O}_{\mathrm{H}} \hookrightarrow \mathcal{O}_{\mathrm{H}}(n) \hookrightarrow \mathrm{G} \otimes \mathcal{O}_{\mathrm{E}}(0,n)$. One verifies that $\mathrm{coker}(s_{n+1}) = \mathcal{O}_{\mathrm{V}}(-1)$.

Lemma 2.4.23. Any BS pair has only the trivial automorphism.

Proof. This is [BS16, Lem. 23]. □

There is a corresponding notion of family of BS pairs.

Definition 2.4.24. Let T be a scheme. A *family of BS pairs parametrised by* T consists of a T-flat sheaf G on $\mathrm{Y} \times \mathrm{T}$ and a section $s \colon \mathcal{O}_{\mathrm{Y}\times\mathrm{T}} \to \mathrm{G}$ such that for all closed points $t \in \mathrm{T}$ the restriction $s_t \colon \mathcal{O}_{\mathrm{Y}} \to \mathrm{G}_t$ is a BS pair. Pullback induces a natural functor

$$f\mathrm{Hilb} \colon \mathrm{Sch}/\mathbf{C} \to \mathrm{Set}, \quad \mathrm{T} \mapsto f\mathrm{Hilb}(\mathrm{T}) = \{\mathrm{T} - \text{families of BS pairs}\}/\sim \tag{2.4.25}$$

from the category of complex schemes to the category of sets. Here the equivalence relation $\sim$ is isomorphism as T-families of BS pairs.

Let $(\beta, n) \in \mathrm{H}_2(\mathrm{Y}, \mathbf{Z}) \oplus \mathbf{Z}$ and let $f\mathrm{Hilb}_{(\beta,n)} \subset f\mathrm{Hilb}$ denote the open and closed subfunctor parametrising BS pairs (G, s) such that $\mathrm{ch}(\mathrm{G}) = (0,0,\beta,n)$. The fundamental result about this functor is the following

Proposition 2.4.25. The functor $f\mathrm{Hilb}_{(\beta,n)}$ has a finite decomposition into locally closed subfunctors each of which is represented by a scheme of finite type.

Proof. This is [BS16, Lem. 23]. □

In particular, BS pairs of class (β, n) form a bounded family.

Remark 2.4.26. In [BS16, Rem. 24] it is conjectured that $f\mathrm{Hilb}_{(\beta,n)}$ is in fact represented by a projective scheme, analogous to the result of PT pairs.

Consider the complement $\mathtt{F}_f := \mathtt{T}_f^{\perp} = \{\mathrm{F} \in \mathrm{Coh}_{\leq 1}(\mathrm{Y}) \mid \mathrm{Hom}(\mathrm{T}, \mathrm{F}) = 0 \text{ for all } \mathrm{T} \in \mathtt{T}_f\}$.

Lemma 2.4.27. The pair $(\mathtt{T}_f, \mathtt{F}_f)$ defines a torsion pair on $\mathrm{Coh}_{\leq 1}(\mathrm{Y})$.

Proof. This is [BS16, Lem. 13]. Indeed, it is easy to see that $\mathtt{T}_f$ is closed under extensions and quotients in $\mathrm{Coh}_{\leq 1}(\mathrm{Y})$ whence Lemma 2.1.17 yields the claim. □

2.5 Rational functions and their re-expansions

It is customary to collect counting invariants in a generating series, whose variables label the numerical data of the objects being enumerated. If these counting invariants have a symmetry, for example induced by a symmetry of the geometric objects they count, this is reflected in nice properties of their generating series.

Throughout, let Y denote a smooth projective Calabi–Yau threefold.

Example 2.5.1. A key example is the rationality of the generating series $\mathrm{PT}(\mathrm{Y})_\beta$ of stable pair invariants on Y of class $\beta \in \mathrm{H}_2(\mathrm{Y}, \mathbf{Z})$. More precisely, $\mathrm{PT}(\mathrm{Y})_\beta$ is the Laurent expansion of a rational function $f_\beta^{\mathrm{Y}}(q)$ around $q = 0$, with the symmetry

$$f_\beta^{\mathrm{Y}}(q) = f_\beta^{\mathrm{Y}}(q^{-1}). \tag{2.5.1}$$

This result was proven by Bridgeland in [Bri11] using ideas of Toda [Tod10b]. The underlying geometric reason for this symmetry is the anti-equivalence

$$\mathbf{D} = \mathbf{R}\underline{\mathrm{Hom}}(-, \mathcal{O}_{\mathrm{Y}})[2] = \underline{\mathrm{Ext}}^2(-, \mathcal{O}_{\mathrm{Y}}) \colon \mathrm{Coh}_1(\mathrm{Y}) \longrightarrow \mathrm{Coh}_1(\mathrm{Y}) \tag{2.5.2}$$

of the full additive subcategory of pure one-dimensional sheaves on Y; see [Bri11, Lem. 5.6]. Crucially, $\mathbf{D}$ flips the sign of the Euler characteristic of the sheaf. This follows since $\mathrm{ch}_i(\mathbf{D}(\mathrm{F})) = (-1)^i \,\mathrm{ch}_i(\mathrm{F})$ for any complex $\mathrm{F} \in \mathrm{D}(\mathrm{Y})$. Recall that the generating series of stable pair invariants on Y of class β is a formal Laurent series defined as

$$\mathrm{PT}(\mathrm{Y})_\beta(q) = \sum_{n \in \mathbf{Z}} \mathrm{PT}_n(\mathrm{Y}, \beta) q^n \in \mathbf{C}((q)). \tag{2.5.3}$$

To answer similar questions for the stable pair generating functions of a CY3 orbifold $\mathcal{X}$, we develop the necessary material on rational generating functions here. First we define generating functions of one or more variables and discuss when they are the

Laurent expansion of a rational function. Then we prove a criterion to decide whether two generating series in reciprocal variables are the Laurent expansion of the same rational function. This result involves quasi-polynomials and is a crucial ingredient of the proof of the crepant resolution conjecture in Chapter 5. Finally, we consider conditions on the coefficients of a *rational* generating series that guarantee the rational function to have certain symmetries.

2.5.1 Generating functions and their rationality

To introduce generating functions and illustrate their utility, we separately treat the univariate and multivariate cases. Some of this material can be found in [Sta97, Chap. 4].

Single variable

Let $\mathbf{C}[x]$ denote the ring of polynomials in one variable with complex coefficients. Its completion at the ideal $(x) \leq \mathbf{C}[x]$, denoted by $\mathbf{C}[[x]]$, is the local ring of formal power series. This ring has as unique maximal ideal (x), whence it follows that an element $\mathrm{A}(x)$ is invertible in $\mathbf{C}[[x]]$ if and only if the constant coefficient $a_0 \neq 0$.

We think of the element $\mathrm{A}(x) \in \mathbf{C}[[x]]$ as the *generating function* of the sequence $\{a_n \in \mathbf{C} \mid n \in \mathbf{Z}\}$, which in turn encodes the values of the function $a\colon \mathbf{Z}_{\geq 0} \to \mathbf{C}$.

Definition 2.5.2. The generating function $\mathrm{A}(x)$ is called *rational* if there exists polynomials $\mathrm{P}(x), \mathrm{Q}(x) \in \mathbf{C}[x]$ with $\mathrm{Q}(0) \neq 0$ such that $\mathrm{A}(x) = \mathrm{P}(x)\mathrm{Q}(x)^{-1}$ in $\mathbf{C}[[x]]$.

The following result is the criterion to recognise rational generating series.

Lemma 2.5.3. Let $q_1, q_2, \ldots, q_d \in \mathbf{C}$ be a fixed sequence of complex numbers, let $d \geq 1$ be an integer, and suppose that $q_d \neq 0$. The following conditions on a function $a\colon \mathbf{Z}_{\geq 0} \to \mathbf{C}$ are equivalent:

1. there exist a polynomial $\mathrm{P}(x) \in \mathbf{C}[x]$ of degree less than d such that
$$\sum_{n \geq 0} a_n x^n = \frac{\mathrm{P}(x)}{\mathrm{Q}(x)},$$
where $\mathrm{Q}(x) = 1 + q_1 x + q_2 x^2 + \ldots + q_d x^d$.
2. For all $n \geq 0$, we have
$$a_{n+d} + q_1 \cdot a_{n+d-1} + q_2 \cdot a_{n+d-2} + \ldots + q_d \cdot a_n = 0. \tag{2.5.4}$$

3. For all $n \geq 0$, there exist non-negative integers d_i such that

$$a_n = \sum_{i=1}^{k} \mathrm{P}_i(n)\gamma_i^n, \tag{2.5.5}$$

where $1 + q_1 x + q_2 x^2 + \ldots + q_d x^d = \prod_{i=1}^{k}(1 - \gamma_i x)^{d_i}$, the γ_i's are distinct, and $\mathrm{P}_i(n)$ is a polynomial in n of degree less than d_i. Note that $d_1 + d_2 + \ldots + d_k \leq d$.

Proof. This is [Sta97, Thm. 4.1.1]. □

Note that the second equivalent condition simply states that the coefficients of the formal power series $\mathrm{Q}(x)\mathrm{A}(x)$ in front of monomials x^n of degree $n \geq d$ vanish, that is, $\mathrm{Q}(x)\mathrm{A}(x) =: \mathrm{P}(x)$ is a *polynomial* of degree less than d. The functions a_n in the generating series one encounters in curve-counting, are typically of a specific type: they are *quasi-polynomial* in the variable $n \in \mathbf{Z}$.

Definition 2.5.4. A function $a: \mathbf{Z} \to \mathbf{C}$ is a quasi-polynomial of quasi-period $p \in \mathbf{Z}_{\geq 1}$ and degree $\leq d$ if there exists a surjective homomorphism $s: \mathbf{Z} \to \mathbf{Z}/p$ such that $a|_{s^{-1}(x)}$ is a polynomial function of degree $\leq d$ for every $x \in \mathbf{Z}/p$.

In words, a quasi-polynomial a of quasi-period $p \in \mathbf{Z}_{\geq 1}$ is a polynomial when restricted to any of the congruence classes of p. That is, there exist p polynomials $a_0, a_1, \ldots, a_{p-1}$ such that

$$a_n = a_i(n) \quad \text{if } n \equiv i \bmod p. \tag{2.5.6}$$

Note that the polynomials a_i will be different in general.

Example 2.5.5. Let $g(n)$ be a polynomial of degree d. Then the function

$$a_n \coloneqq (-1)^n g(n) = \begin{cases} g(n) & \text{if } 2 \mid n \\ -g(n) & \text{if } 2 \nmid n \end{cases} \tag{2.5.7}$$

is a quasi-polynomial of quasi-period 2, and degree d.

The statements of Lemma 2.5.3 can be strengthened when a is a quasi-polynomial.

Lemma 2.5.6. The following conditions on a function $a: \mathbf{Z}_{\geq 0} \to \mathbf{C}$ and an integer $p > 0$ are equivalent:

1. a is a quasi-polynomial of quasi-period p, and
2. there exist $\mathrm{P}(x), \mathrm{Q}(x) \in \mathbf{C}[x]$ such that $\gcd(\mathrm{P}, \mathrm{Q}) = 1$ and

$$\sum_{n \geq 0} a_n x^n = \frac{\mathrm{P}(x)}{\mathrm{Q}(x)},$$

where every zero α of $\mathrm{Q}(x)$ satisfies $\alpha^p = 1$, and $\deg(\mathrm{P}) < \deg(\mathrm{Q})$.

3. For all $n \geq 0$, we have
$$a_n = \sum_{i=1}^{k} \mathrm{P}_i(n)\gamma_i^n, \tag{2.5.8}$$
where each P_i is a polynomial function of n and each γ_i satisfies $\gamma_i^p = 1$.

Proof. This is [Sta97, Prop. 4.4.1]. □

To summarise, a generating function in one variable x with quasi-polynomial coefficients a_n is the Taylor expansion about $x = 0$ of a rational function with poles at roots of unity, determined by the quasi-period of a_n.

Multiple variables

The generating series of curve-counting invariants on the orbifold $\mathcal{X}$ are series in multiple variables, essentially because the rank of the numerical Grothendieck group $\mathrm{N}_0(\mathcal{X})$ is larger than one, as in the case of smooth varieties; see Example 2.1.10. Moreover, these series are often *rational*. To have well-defined expansions in multiple variables,
$$\mathrm{A}(q) = \sum_{c \in \mathrm{N}_0(\mathcal{X})} a(c)q^c, \tag{2.5.9}$$
restrictions must be imposed on the sets $\{c \in \mathrm{N}_0(\mathcal{X}) \mid a(c) \neq 0\}$ appearing. These can be neatly phrased in terms of various notions of boundedness of subsets of $\mathrm{N}_0(\mathcal{X})$.

To keep the discussion general, let Γ denote a free abelian group of finite rank.

Definition 2.5.7. Let $\mathrm{L}\colon \Gamma \to \mathbf{R}$ be a group homomorphism. We say a set $\mathrm{S} \subset \Gamma$ is L-*bounded* if $\mathrm{S} \cap \{c \in \Gamma \mid \mathrm{L}(c) \leq \mathrm{M}\}$ is finite for every $\mathrm{M} \in \mathbf{R}$.

We say S is *weakly* L-*bounded* if the image of S in $\Gamma/\ker(\mathrm{L})$ is L-bounded.

For a fixed homomorphism $\mathrm{L}\colon \Gamma \to \mathbf{R}$, we have two easy properties and a definition.

Lemma 2.5.8. Let S and T be (weakly) L-bounded sets.

1. A finite union of (weakly) L-bounded sets is again (weakly) L-bounded.

2. The sum $\mathrm{S} + \mathrm{T} = \{s + t \mid s \in \mathrm{S}, t \in \mathrm{T}\}$ is again (weakly) L-bounded.

Definition 2.5.9. Let $\mathbf{Z}\{\Gamma\}$ be the additive group of all infinite formal sums of terms $a(c)q^c$ with $a(c) \in \mathbf{Z}$, and $\mathbf{Z}[\Gamma]$ the additive group of all finite such sums. We define $\mathbf{Z}[\Gamma]_{\mathrm{L}} \subset \mathbf{Z}\{\Gamma\}$ to be the subset of those formal sums for which $\{c \in \Gamma \mid a(c) \neq 0\}$ is L-bounded.

Corollary 2.5.10. By the lemma, $\mathbf{Z}[\Gamma]_{\mathrm{L}}$ is a ring under the obvious operations.

Remark 2.5.11. The product of two elements in $\mathbf{Z}\{\Gamma\}$ is in general not well defined. However, multiplication by an element $f \in \mathbf{Z}[\Gamma]$ induces a well-defined linear action

$$\mathrm{M}\colon \mathbf{Z}[\Gamma] \times \mathbf{Z}\{\Gamma\} \to \mathbf{Z}\{\Gamma\}, \quad (f, \mathrm{T}) \mapsto \mathrm{M}_f(\mathrm{T}) \coloneqq f \cdot \mathrm{T}. \tag{2.5.10}$$

This equips $\mathbf{Z}\{\Gamma\}$ with a natural structure of $\mathbf{Z}[\Gamma]$-module.

Definition 2.5.12. Given a rational function $f = g/h$ with $g, h \in \mathbf{Z}[\Gamma]$, we say that a series $f_{\mathrm{L}} \in \mathbf{Z}[\Gamma]_{\mathrm{L}}$ is the expansion of f in $\mathbf{Z}[\Gamma]_{\mathrm{L}}$ if $f_{\mathrm{L}} h = g$ holds in the ring $\mathbf{Z}[\Gamma]_{\mathrm{L}}$.

Note that such an expansion may not exist for all choices of f and L, but if it does, it is unique.

Example 2.5.13. Consider the rational function $f(x) = x/(1+x)^2$, that appeared as the re-summation of a PT generating function on local $\mathbf{P}^1$ in Example 1.2.2; note that $x, (1+x)^2 \in \mathbf{Z}[x]$. Let $\mathrm{L}_+\colon \mathbf{Z} \to \mathbf{R}$ be the group homomorphism $\mathrm{L}_+(k) = k$. Then

$$f_{\mathrm{L}_+}(x) = \sum_{n=0}^{\infty} (-1)^{n-1} n x^n, \tag{2.5.11}$$

and we have $(1+x)^2 f_{\mathrm{L}_+}(x) = x$ in $\mathbf{Z}[x]_{\mathrm{L}_+}$ as the reader easily verifies. The choice of homomorphism L_+ means we are expanding the rational function f around $x = 0$.

As in the single variable case, there is a notion of quasi-polynomial coefficients.

Definition 2.5.14. A function $a\colon \Gamma \to \mathbf{C}$ is said to be a *quasi-polynomial* of *quasi-period* p and degree $\leq d$ if there exists a surjective homomorphism $s\colon \Gamma \to \Gamma/p\Gamma$ such that $a|_{s^{-1}(x)}$ is a polynomial function of degree $\leq d$ for every $x \in \Gamma/p\Gamma$.

The following is the required result on quasi-polynomial generating functions.

Lemma 2.5.15. Let $r \in \mathbf{Z}_{\geq 1}$, let $\mathrm{E} \subset \{1, \ldots, r-1\}$, and let $a\colon \Gamma = \mathbf{Z}^r \to \mathbf{C}$ be a quasi-polynomial in r variables of quasi-period p. Consider the generating series

$$\mathrm{P}(x_1, \ldots, x_r) = \sum_{n_1, \ldots, n_r} a(n_1, \ldots, n_r) x_1^{n_1} \cdots x_r^{n_r}, \tag{2.5.12}$$

where the sum runs over all sequences of integers

$$\{(n_1, \ldots, n_r) \in \mathbf{Z}^r \mid 0 \leq n_1 \leq \ldots \leq n_r, \text{ and } n_i = n_{i+1} \text{ iff } i \in \mathrm{E}\}.$$

Then P is a series expansion of the rational function g/h, where g is a polynomial in the variables x_i, and $h = \prod_{i \in \{0, \ldots, r-1\} \setminus \mathrm{E}} (1 - \prod_{j=i+1}^{r} q_j^p)^{e_i}$, where $e_i = 1 + \sum_{j=i+1}^{r} \deg_{n_i}(a)$.

Proof. Set $n_0 = 0$. Let $k_i = n_i - n_{i-1}$ and rewrite the claim in terms of the k_i. So

$$\mathrm{P} = \sum_{i \notin \mathrm{E}: k_i \geq 0} a(k_1, k_1 + k_2, \ldots, k_1 + \ldots + k_r) \prod_{j=1}^{r} (x_j \ldots x_r)^{k_j},$$

where we read $k_{i+1} = 0$ if $i \in \mathrm{E}$. Note that a (quasi-)polynomial of degree k yields a denominator of the form $(-)^{k+1}$. If a is a polynomial, the result follows since we recognise the geometric series and its derivatives. If a is a quasi-polynomial of quasi-period p, the above is a sum of p^r such polynomial cases. This completes the proof. □

2.5.2 Re-expanding rational functions

The following result allows us to detect when two generating series are the Laurent expansion of the *same* rational function in *reciprocal* variables.

Lemma 2.5.16. Suppose we have $0 \neq c_0 \in \Gamma$, and linear functions $\mathrm{L}_-, \mathrm{L}_+ : \Gamma \to \mathbf{R}$ such that $\mathrm{L}_-(c_0) < 0$ and $\mathrm{L}_+(c_0) > 0$. Let f be a rational function admitting an expansion f_{L_-} in $\mathbf{Z}[\Gamma]_{\mathrm{L}_-}$, and let $f' \in \mathbf{Z}[\Gamma]_{\mathrm{L}_+}$ be a function such that for any $c \in \Gamma$, the coefficient of q^{c+kc_0} in $f_{\mathrm{L}_-} - f'$ is a quasi-polynomial in $k \in \mathbf{Z}$. Then $f' = f_{\mathrm{L}_+}$ in $\mathbf{Z}[\Gamma]_{\mathrm{L}_+}$.

Proof. Write $f = g/h$ for polynomials $g, h \in \mathbf{Z}[\Gamma]$. Consider the object $(f_{\mathrm{L}_-} - f')h$. This has the property that the coefficient of q^{c+kc_0} is quasi-polynomial in k for any $c \in \Gamma$, since $f_{\mathrm{L}_-} - f'$ has this property and h is a polynomial.

If we let $k \to -\infty$, the coefficient of q^{c+kc_0} in $f'h$ is 0 since $f' \in \mathbf{Z}[\Gamma]_{\mathrm{L}_+}$ and $\mathrm{L}_+(c_0) > 0$. On the other hand, the coefficient of q^{c+kc_0} in $f_{\mathrm{L}_-}h$ will be 0 since $f_{\mathrm{L}_-}h = g$ in $\mathbf{Z}[\Gamma]_{\mathrm{L}_-}$ which is a polynomial. By quasi-polynomiality of the coefficients of the difference $(f_{\mathrm{L}_-} - f')h$, it follows that each quasi-polynomial coefficient is 0. Thus $(f_{\mathrm{L}_-} - f')h = 0$ identically in $\mathbf{Z}\{\Gamma\}$.

Since $f_{\mathrm{L}_-}h = g$ it follows that $f'h = g$. But $f' \in \mathbf{Z}[\Gamma]_{\mathrm{L}_+}$ and h is a polynomial, so $f'h = g$ holds in $\mathbf{Z}[\Gamma]_{\mathrm{L}_+}$. We conclude that $f' = f_{\mathrm{L}_+}$ is the re-expansion as claimed. □

Example 2.5.17. Consider again the rational function $f(x) = x/(1+x)^2$, with $g = x$ and $h = (1+x)^2$, as in the previous example. Then

$$(1+x)^2 f_{\mathrm{L}_+}(x) = \left(\sum_{n=0}^{\infty} (-1)^{n-1} n x^n \right) (1+x)^2 = x \quad \text{in} \quad \mathbf{Z}[x]_{\mathrm{L}_+}.$$

Alternatively, we may re-expand f with respect to $\mathrm{L}_- : \mathbf{Z} \to \mathbf{R}$, $\mathrm{L}_-(k) = -k$. Because of the symmetry $f(x) = f(x^{-1})$, our educated guess for the expansion of f in $\mathbf{Z}[x]_{\mathrm{L}_-}$ is

$$f'(x) = \sum_{n=0}^{\infty} (-1)^{n-1} n x^{-n}. \tag{2.5.13}$$

Indeed, $f' \in \mathbf{Z}[x]_{\mathrm{L}_-}$. To apply the previous lemma take $c_0 = 1$, so $\mathrm{L}_+(c_0) > 0$ and $\mathrm{L}_-(c_0) < 0$. The coefficient of the term x^{c+kc_0} in the difference

$$f_{\mathrm{L}_+}(x) - f'(x) = \sum_{n=-\infty}^{\infty} (-1)^{n-1} n x^n \tag{2.5.14}$$

is equal to $(-1)^{c+kc_0-1}(c+kc_0) = (-1)^{c+k-1}(c+k)$ for all $c, k \in \mathbf{Z}$ since $c_0 = 1$. This is a quasi-polynomial of quasi-period $p = 2$. Hence

$$f'(x)(1+x)^2 = \left(\sum_{n=0}^{\infty} (-1)^{n-1} x^{-n}\right)(1+x)^2 = x \quad \text{in} \quad \mathbf{Z}[x]_{\mathrm{L}_-}$$

as the reader easily verifies, and we may conclude that $f' = f_{\mathrm{L}_-}$. Note that the choice of homomorphism L_- means we are expanding the rational function f around $x = \infty$. The root α of the denominator of the rational function f indeed satisfies $\alpha^p = 1$.

Remark 2.5.18. Since $f_{\mathrm{L}_\pm} h = g$ in $\mathbf{Z}[x]_{\mathrm{L}_\pm}$ and $f' = f_{\mathrm{L}_-}$, lemma 2.5.16 has the surprising interpretation that we should think of the function in (2.5.14) as the zero function.

However, this is not quite accurate. Indeed, multiplying an infinite formal power series in $\mathbf{Z}\{x\}$ by a Laurent polynomial is a well-defined linear operation as was observed in Remark 2.5.11. In this case, we are merely observing the fact that

$$(1+x)^2 \cdot \left(\sum_{n=-\infty}^{\infty} (-1)^{n-1} n x^n\right) = 0 \quad \text{in } \mathbf{Z}\{x\}. \tag{2.5.15}$$

A similar analysis on the rational function $1/(1-x)$ yields

$$(1-x) \cdot \left(\sum_{n=-\infty}^{\infty} x^n\right) = 0 \quad \text{in } \mathbf{Z}\{x\}. \tag{2.5.16}$$

In this example, it was pointed out to us by A. Okounkov that $\sum_{n=-\infty}^{\infty} x^n$ is the Fourier series of the Dirac delta function at the point $1 \in \mathbf{C}$ upon setting $x = e^{i\pi}$. When one multiplies this Dirac delta function by $(1-x)$ that function, indeed, vanishes.

Chapter 3

A counterexample

Recall the setting of the crepant resolution conjecture in Section 2.4. We have a smooth CY3 orbifold $\mathcal{X}$ satisfying the hard Lefschetz condition with projective coarse moduli space $\pi\colon \mathcal{X} \to \mathrm{X}$. Results of [BKR01, CT08] provide us with a natural crepant resolution $f\colon \mathrm{Y} \to \mathrm{X}$. The situation is summarised in the following diagram.

$$\begin{tikzcd} \mathcal{X} \arrow[dr, "\pi"'] & & \mathrm{Y} \arrow[dl, "f"] \\ & \mathrm{X} & \end{tikzcd} \tag{3.0.1}$$

The McKay equivalence $\Phi\colon \mathrm{D(Y)} \to \mathrm{D}(\mathcal{X})$ of [BKR01] sends $\Phi(\mathcal{O}_\mathrm{Y}) = \mathcal{O}_\mathcal{X}$ and induces an identification of the Grothendieck groups of the orbifold $\mathcal{X}$ and the resolution Y. This identification descends to the numerical Grothendieck groups, and we denote this linear isomorphism by $\phi\colon \mathrm{N}(\mathcal{X}) \to \mathrm{N(Y)}$. If the crepant resolution conjecture is to hold as an equality of generating series, then there cannot exist classes α on the orbifold such that

$$\left.\frac{\mathrm{DT}(\mathcal{X})}{\mathrm{DT}_0(\mathcal{X})}\right|_{\phi(\alpha)} \neq 0 \quad \text{whereas} \quad \left.\frac{\mathrm{DT(Y)}}{\mathrm{DT(Y)_{exc}}}\right|_{\alpha} = 0, \tag{3.0.2}$$

or vice versa, since the DT generating series would not be equal term-by-term. Here $\mathrm{F}|_\alpha$ denotes the α coefficient in the generating series F.

In this chapter, we show that this mismatch already occurs in the simplest geometry to which the conjecture applies; consequently, the crepant resolution conjecture does not hold in general as an equality of generating series. This is achieved by explicitly computing the Donaldson–Thomas invariants of certain classes. As a corollary, we observe that the corresponding generating series are Laurent expansions of the *same* rational function (a geometric series) at different points. This suggests one should reinterpret the crepant resolution conjecture as an equality of rational functions, suitable expansions of which encode the Donaldson–Thomas invariants of $\mathcal{X}$ and Y.

3.1 The stacky local projective line

Our counterexample is a certain total space on the stacky local projective line constructed as follows. Let T be the total space $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1$, and let $\mathbf{Z}_2$ act fibre-wise on T by sending fibre coordinates $(x, y) \mapsto (-x, -y)$. Let $\mathcal{X} = [\mathrm{T}/\mathbf{Z}_2]$ be the stacky quotient, let $\mathrm{X} = \mathrm{T}/\mathbf{Z}_2$ be the usual quotient, and let $f\colon \mathrm{Y} \to \mathrm{X}$ be the natural crepant resolution of singularities given by [BKR01]. This is a trivial $\mathbf{P}^1$-family of A_1-surface singularities.

The crepant resolution Y of the coarse moduli space X is the fibre-wise resolution of these surface singularities. Hence Y may be identified with the total space of $\mathcal{O}(-2,-2) \to \mathbf{P}^1 \times \mathbf{P}^1$, and f contracts its zero-section $\mathrm{pr}_1\colon \mathrm{E} = \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ fibre-wise.

Remark 3.1.1. Note that the coarse moduli space X is merely *quasi-projective*, not projective as in the statement of the crepant resolution conjecture 5.1.1. The modified conjecture (an equality of rational functions) is expected to hold in this case as well, but we remark that it is essential that the underlying smooth curve be $\mathbf{P}^1$. See [BCR].

Given the structure sheaf $\mathcal{O}_\mathrm{Z}$ of a proper subscheme $\mathrm{Z} \subset \mathcal{X}$, we let $\mathcal{O}_\mathrm{Z}^+$ (resp. $\mathcal{O}_\mathrm{Z}^-$) denote the trivial (resp. non-trivial) $\mathbf{Z}_2$-equivariant structure on $\mathcal{O}_\mathrm{Z}$. Furthermore, we write C for the $\mathbf{P}^1$ in $\mathcal{X}$. Since Z is assumed proper, its set-theoretical support lies in C.

Let $\Phi\colon \mathrm{D}(\mathrm{Y}) \to \mathrm{D}(\mathcal{X})$ denote the McKay correspondence, and recall $\Phi(\mathcal{O}_\mathrm{Y}) = \mathcal{O}_\mathcal{X}$. If $p \in \mathrm{X}$ is a point in the singular locus, then $f_p = f^{-1}(p) \subset \mathrm{Y}$ denotes its fibre in Y. Furthermore, we simply write $p \in \mathcal{X}$ if no confusion can occur.

Lemma 3.1.2. The McKay correspondence acts as follows on some objects of $\mathrm{D}(\mathrm{Y})$:

$$\begin{aligned} \Phi(\mathcal{O}_{f_p}(-2)[1]) &= \mathcal{O}_p^+ \qquad & \Phi(\mathcal{O}_\mathrm{E}(-1,-2)[1]) &= \mathcal{O}_\mathrm{C}(-1)^+ \\ \Phi(\mathcal{O}_{f_p}(-1)) &= \mathcal{O}_p^- \qquad & \Phi(\mathcal{O}_\mathrm{E}(-1,-1)) &= \mathcal{O}_\mathrm{C}(-1)^- \end{aligned} \tag{3.1.1}$$

Proof. This is a $\mathbf{P}^1$-family version of the surface computation in Example 2.4.10. In particular, the left two equations for $\mathcal{O}_p^\pm$ have been determined there. Since X is a *trivial* $\mathbf{P}^1$-family of the surface case, the right two equations follow. □

Recall the definition in section 2.1.7 of the numerical Grothendieck group $\mathrm{N}_{\leq 1}(\mathcal{X})$ generated by sheaves of at most one-dimensional support on $\mathcal{X}$. In the current geometry, it has the following natural generators

$$\mathrm{N}_{\leq 1}(\mathcal{X}) = \langle \mathcal{O}_\mathrm{C}^+(-1), \mathcal{O}_\mathrm{C}^-(-1), \mathcal{O}_p^+, \mathcal{O}_p^- \rangle.$$

Note that $\chi(\mathcal{X}, \mathcal{O}_p^+) = 1$ whereas the other generators have vanishing Euler characteristic. We write $s^\pm = [\mathcal{O}_\mathrm{C}^\pm(-1)]$ and $p^\pm = [\mathcal{O}_p^\pm]$ for simplicity. We need a set of generators for $\mathrm{N}_{\mathrm{mr}}(\mathcal{X}) = \Phi(\mathrm{N}_{\leq 1}(\mathrm{Y}))$. We set $e_s = s^+ + s^-$, $e_f = p^-$, and $e_p = p^+ + p^-$. One easily verifies that $\mathrm{N_{mr}}(\mathcal{X}) = \langle e_s, e_f, e_p\rangle$ freely generates, so that every multi-regular class $\alpha \in \mathrm{N_{mr}}(\mathcal{X})$ can be uniquely written as

$$\alpha = s(\alpha)e_s + f(\alpha)e_f + p(\alpha)e_p, \tag{3.1.2}$$

where $s, f, p \in \mathrm{N_{mr}}(\mathcal{X})^\vee$ denote the dual basis. Moreover it follows from (3.1.1) that

$$\Phi([\mathcal{O}_\mathrm{H}(-1)]) = e_s, \quad \Phi([\mathcal{O}_{f_p}(-1)]) = e_f, \quad \Phi([\mathcal{O}_y]) = e_p,$$

where $\mathrm{H} \subset \mathrm{E}$ is a horizontal section of the exceptional locus E of f, and $y \in \mathrm{Y}$ is a closed point. Thus we may use the classes $\{e_s, e_f, e_p\}$ as a basis for the curve classes in $\mathrm{N_{mr}}(\mathcal{X}) = \Phi(\mathrm{N}_{\le 1}(\mathrm{Y}))$, with a clear geometric interpretation on both sides.

3.2 Computing Donaldson–Thomas invariants

We exhibit a family of numerical classes on $\mathcal{X}$ and Y, identified by ϕ, and compute the corresponding set of DT invariants, such that the mismatch of equation (3.0.2) occurs. The key remark is that the class $(2,0,2)$ is the class of a quotient of $\mathcal{O}_\mathrm{Y}$ in $\mathrm{Coh}(\mathrm{Y})$ but it is *not* the class of a quotient of $\mathcal{O}_\mathcal{X}$ in $\mathrm{Coh}(\mathcal{X})$; we say that $(2,0,2)$ is an *effective curve class* or *quotient class* on Y but not on $\mathcal{X}$.

First, we introduce notation to distinguish three notions of numerical classes. We do so in terms of Y, but the same terminology will be used for $\mathcal{X}$.

1. A *numerical class* is a class $\alpha \in \mathrm{N}(\mathrm{Y})$.
2. A *(numerically) effective class* is a numerical class of a sheaf on Y, i.e., there exists a sheaf $\mathrm{F} \in \mathrm{N}(\mathrm{Y})$ such that $\alpha = [\mathrm{F}]$; we write $\mathrm{NE}(\mathrm{Y}) \subset \mathrm{N}(\mathrm{Y})$ for the convex *numerically effective cone* generated by effective classes.
3. A *quotient class* is an effective class $\alpha \in \mathrm{N}_{\le 1}(\mathrm{Y})$ represented by a curve, i.e., there exists a one-dimensional quotient $\mathcal{O}_\mathrm{Y} \twoheadrightarrow \mathcal{O}_\mathrm{C}$ such that $\alpha = [\mathcal{O}_\mathrm{C}]$; we write $\mathrm{Eff}(\mathrm{Y}) \subset \mathrm{NE}(\mathrm{Y})$ for the convex *quotient cone* generated by quotient classes.

We compute both sides of (3.0.2) for all quotient classes of the form $(2, f, 2-f)$ where $f \in \mathbf{Z}$. Note that such a class can be a quotient class on Y but not on $\mathcal{X}$ and vice versa. These computations will allow us to conclude. Note that the quotients on both sides of equation (3.0.2) are a complicating factor and should be treated with care.

First, we plot the intersection of the quotient cone of curves of $\mathcal{X}$ with the plane $\{s = 2\}$ inside $\mathrm{N_{mr}}(\mathcal{X})$. These are classes α with $s(\alpha) = 2$ and $\mathrm{Hilb}_\mathcal{X}(\alpha) \neq \varnothing$. Since our computation will also involve (parts of) the series $\mathrm{DT}(\mathrm{Y})_\mathrm{exc}, \mathrm{DT}(\mathrm{Y})^\vee_\mathrm{exc}$, and $\mathrm{DT}(\mathcal{X})_0$ we plot the corresponding quotient cones $\mathrm{Eff}_\mathrm{exc}(\mathrm{Y})$ and $\mathrm{Eff}^\vee_\mathrm{exc}(\mathrm{Y})$, and the entire numerically effective cone $\mathrm{NE}_0(\mathcal{X})$, even though these lie in the plane $\{s = 0\}$. A class

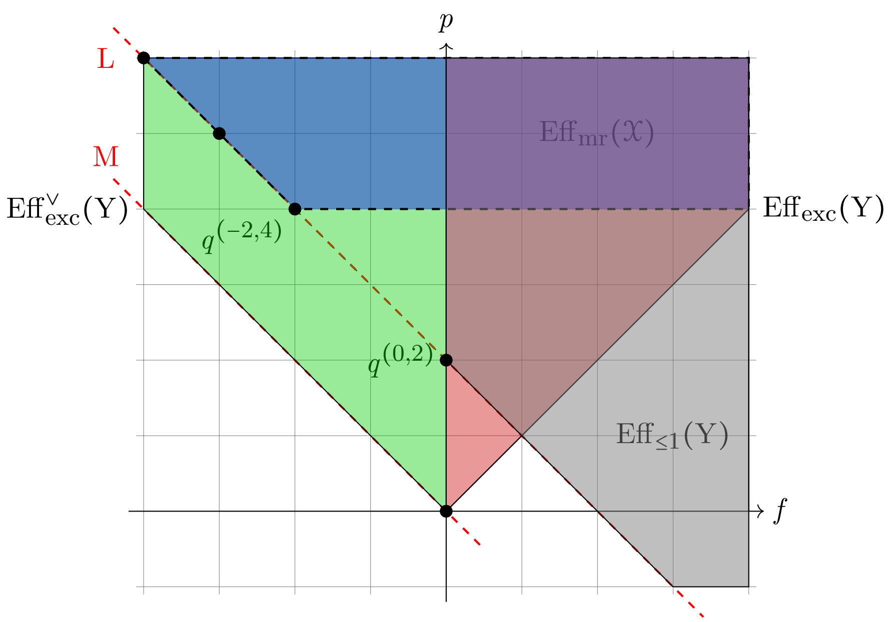


Figure 3.1: A slice at $s = 2$ of the numerically effective cones of $\mathcal{X}$ and Y; see Lemma 3.2.1.

α with $s(\alpha) = 2$ is drawn with coordinates $(f(\alpha), p(\alpha))$, i.e., its fibre and point class respectively. The determine the relevant cones.

Lemma 3.2.1. We have the following quotient and numerically effective cones of $\mathcal{X}$ and Y in the plot, in terms of their generators, both explicitly and in coordinates:

1. $\mathrm{Eff}_{\mathrm{mr}}(\mathcal{X}) = 2(e_s - e_f + 2e_p) + \mathbf{Z}_{\geq 0} \cdot (-e_f + e_p) \oplus \mathbf{Z}_{\geq 0} \cdot e_p = (2, -2, 4) + \langle (0, -1, 1), (0, 0, 1) \rangle$;
2. $\mathrm{Eff}_{\leq 1}(\mathrm{Y}) = (2e_s + 2e_p) + \mathbf{Z}_{\geq 0} \cdot (e_f - e_p) \oplus \mathbf{Z}_{\geq 0} \cdot e_p = (2, 0, 2) + \langle (0, 1, -1), (0, 0, 1) \rangle$;
3. $\mathrm{Eff}_{\mathrm{exc}}(\mathrm{Y}) = \mathbf{Z}_{\geq 0} \cdot (e_f + e_p) \oplus \mathbf{Z}_{\geq 0} \cdot e_p = \langle (0, 1, 1), (0, 0, 1) \rangle$;
4. $\mathrm{Eff}^{\vee}_{\mathrm{exc}}(\mathrm{Y}) = \mathbf{Z}_{\geq 0} \cdot (-e_f + e_p) \oplus \mathbf{Z}_{\geq 0} \cdot e_p = \langle (0, -1, 1), (0, 0, 1) \rangle$;
5. $\mathrm{Eff}_0(\mathcal{X}) = \mathbf{Z}_{\geq 0} \cdot (-e_f + e_p) \oplus \mathbf{Z}_{\geq 0} \cdot e_p \oplus \mathbf{Z}_{\geq 0} \cdot (e_f + e_p) = \langle (0, -1, 1), (0, 0, 1), (0, 1, 1) \rangle$.
6. $\mathrm{NE}_0(\mathcal{X}) = \mathbf{Z}_{\geq 0} \cdot (-e_f + e_p) \oplus \mathbf{Z}_{\geq 0} \cdot e_f = \langle (0, -1, 1), (0, 1, 0) \rangle$.

Note that the subcones $\mathrm{Eff}_0(\mathcal{X}) \subset \mathrm{NE}_0(\mathcal{X})$, which would be drawn as the union of $\mathrm{N}_{\mathrm{exc}}(\mathrm{Y})$ and $\mathrm{N}^{\vee}_{\mathrm{exc}}(\mathrm{Y})$, are not depicted, and neither is the p-axis $\mathrm{NE}_0(\mathrm{Y}) = \langle (0, 0, 1) \rangle$.

Proof. We have $\mathrm{N}_0(\mathcal{X}) = \mathbf{Z} \cdot [\mathcal{O}_p^+] \oplus \mathbf{Z} \cdot [\mathcal{O}_p^-]$, and only $\mathcal{O}_p^+$ and $\mathcal{O}_p \otimes \mathbf{C}[\mathbf{Z}_2]$ are quotients of $\mathcal{O}_{\mathcal{X}}$. This yields the description of $\mathrm{Eff}_0(\mathcal{X}) \subset \mathrm{NE}_0(\mathcal{X})$. Similarly, $\mathcal{O}_y$ for $y \in \mathrm{Y}$ and the structure sheaf of a fibre $\mathcal{O}_{f_p}$ induce the exceptional quotient classes on Y. Since $[\mathcal{O}_{f_p}] = [\mathcal{O}_{f_p}(-1)] + [\mathcal{O}_p]$ in $\mathrm{N}(\mathrm{Y})$, this explains $\mathrm{Eff}_{\mathrm{exc}}(\mathrm{Y})$ and, by duality, $\mathrm{Eff}^{\vee}_{\mathrm{exc}}(\mathrm{Y})$.

Next we describe the multi-regular quotients classes on $\mathcal{X}$, i.e., $\mathrm{Eff}_{\mathrm{mr}}(\mathcal{X})$. First, we explain the term $2e_s - 2e_f + 4e_p$. A quotient $\mathcal{O}_\mathrm{Z}$ of $\mathcal{O}_\mathcal{X}$ is equivalent to a quotient of the sheaf of $\mathbf{Z}_2$-equivariant $\mathcal{O}_{\mathbf{P}^1}$-algebras

$$\begin{aligned} \mathrm{B} &:= \bigoplus_{n\geq 0} \mathcal{O}^+_\mathrm{C}(2n)^{\oplus 2n+1} \oplus \mathcal{O}^-_\mathrm{C}(2n+1)^{\oplus 2n+2} \\ &= \mathcal{O}^+_\mathrm{C} \oplus \mathcal{O}^-_\mathrm{C}(1)^{\oplus 2} \oplus \mathcal{O}^+_\mathrm{C}(2)^{\oplus 3} \oplus \mathcal{O}^-_\mathrm{C}(3)^{\oplus 4} \oplus \dots \end{aligned} \tag{3.2.1}$$

where C denotes the zero section and the superscripts $\pm$ denote the trivial and non-trivial $\mathbf{Z}_2$-equivariant structure respectively. Any multi-regular curve of class α with $s(\alpha) = 2$ is a quotient of B containing two copies of either representation; note that the quotient is taken in the category of coherent sheaves of B-modules.

Consider a quotient $\mathrm{B} \twoheadrightarrow \pi_*\mathcal{O}_\mathrm{Z} \twoheadrightarrow \pi_*\mathcal{O}_{\overline{\mathrm{Z}}}$ where $\pi\colon \mathcal{X} \to [\mathbf{P}^1/\mathbf{Z}_2]$ denotes the structure morphism, and $\overline{\mathrm{Z}} \subset \mathrm{Z}$ is the maximal pure one-dimensional subcurve. By the $\mathbf{Z}_{\geq 0}$-graded algebra structure of B, if $(\pi_*\mathcal{O}_{\overline{\mathrm{Z}}})_{\geq 4} \neq 0$ then this implies that $(\pi_*\mathcal{O}_{\overline{\mathrm{Z}}})_i \neq 0$ for $i = 0,1,2,3$. But this contradicts $\mathrm{rk}\,\pi_*\mathcal{O}_{\overline{\mathrm{Z}}} = 4$ so $\mathrm{B}_{\leq 3} \twoheadrightarrow (\pi_*\mathcal{O}_{\overline{\mathrm{Z}}})$ is surjective in degree at most three.

Consider the induced exact sequence of sheaves of $\mathbf{Z}_2$-equivariant B-modules

$$0 \to \mathrm{I} \to \mathrm{B} \xrightarrow{p} \pi_*\mathcal{O}_{\overline{\mathrm{Z}}} \to 0.$$

We distinguish two cases:

1. Suppose $\mathcal{O}^-_\mathrm{C}(1)^{\oplus 2} \hookrightarrow \pi_*\mathcal{O}_{\overline{\mathrm{Z}}}$. By the algebra structure, it follows that

$$\mathcal{O}^+_\mathrm{C} \oplus \mathcal{O}^-_\mathrm{C}(1)^{\oplus 2} \oplus \mathcal{O}^+_\mathrm{C}(2) \hookrightarrow \pi_*\mathcal{O}_{\overline{\mathrm{Z}}} \tag{3.2.2}$$

   and, hence, $\mathrm{I}_3 \hookrightarrow \mathcal{O}^-_\mathrm{C}(3)^{\oplus 4}$. It follows that $[\pi_*\mathcal{O}_{\overline{\mathrm{Z}}}] \in (2,0,4) + \mathrm{NE}_0(\mathcal{X})$ by a computation of the numerical class of the injecting sheaf, which lies above the extremal ray defined by $\mathrm{L} = \{(f, 2-f) \mid f \in \mathbf{Z}\}$ as required.

2. Suppose that p is not injective in degree 1. The algebra structure then forces

$$\mathcal{O}^+_\mathrm{C} \oplus \mathcal{O}^-_\mathrm{C}(1) \oplus \mathcal{O}^+_\mathrm{C}(2) \oplus \mathcal{O}^-_\mathrm{C}(3) \hookrightarrow \pi_*\mathcal{O}_{\overline{\mathrm{Z}}}. \tag{3.2.3}$$

   But then $[\pi_*\mathcal{O}_{\overline{\mathrm{Z}}}] \in (2,-2,4) \oplus \mathrm{NE}_0(\mathcal{X})$.

Since $\mathrm{NE}_0(\mathcal{X}) = \langle (0,-1,1), (0,1,0) \rangle$, this completes the claim for $\mathrm{Eff}_{\mathrm{mr}}(\mathcal{X})$.

As for $\mathrm{Eff}_{\leq 1}(\mathrm{Y})$, write $\mathrm{E} = \mathbf{P}^1 \times \mathbf{P}^1$. A similar reasoning shows that a quotient $\mathcal{O}_\mathrm{Z}$

of $\mathcal{O}_Y$ is equivalent to a quotient of the sheaf of $\mathcal{O}_E$-algebras

$$B_E := \bigoplus_{n \geq 0} \mathcal{O}_E(2n, 2n) = \mathcal{O}_E \oplus \mathcal{O}_E(2,2) \oplus \mathcal{O}_E(4,4) \oplus \dots \tag{3.2.4}$$

The quotient with minimal Euler characteristic is the zero locus Z of a section of $\mathcal{O}_E(2, b)$. Thus $b \geq 2$, and we compute $\chi(\mathcal{O}_Z) = 2 - b$. Since $\mathrm{Eff}_0(Y)$ is generated by $e_p = [\mathcal{O}_y]$ for $y \in Y$, the claim now follows. □

The classes of interest lie on the extremal ray $L\colon p = 2 - f$ of the cone $\mathrm{NE}_{\leq 1}(Y)$ restricted to the plane $s = 2$. We now proceed to compute their DT invariants.

Lemma 3.2.2. Set $x = q^{(1,-1)}$, and consider the above setting and notation.

1. For the invariants on the resolution, we have

$$\mathrm{DT}_Y(2, f, 2-f) = 3(-1)^f(f+1)$$

   for $f \geq 0$, whereas the invariants vanish for $f < 0$.

2. For the invariants on the orbifold, we have

$$\begin{aligned} \mathrm{DT}_{\mathcal{X}}(0, -f, f) &= (-1)^f(f+1) \\ \mathrm{DT}_{\mathcal{X}}(2, -2-f, 4+f) &= (-1)^f(f+1)(f+2)(f+3)/2 \end{aligned}$$

   for $f \geq 0$, whereas the invariants vanish for $f < 0$.

Proof. Recall that if M is a smooth scheme of finite type, then its Behrend weighted Euler characteristic simply equals $e_B(M) = (-1)^m e(M)$, where $m = \dim(M)$. A curve of class $(a, b, ab + a + b)$ on Y corresponds to the zero locus of a section of $\mathcal{O}_E(a, b)$. The Hilbert scheme of curves of class $(2, f, 2-f)$ on Y can be identified with the smooth variety $\mathrm{Hilb}_Y(2, f, 2-f) = \mathbf{P}H^0(E, \mathcal{O}_E(f, 2))^\vee$. It follows that

$$\mathrm{DT}_Y(2, f, 2-f) = (-1)^{2+f+2f}(2+1)(f+1) = 3(-1)^f(f+1) \tag{3.2.5}$$

as claimed. The full zero-dimensional partition function is $\mathrm{DT}(Y)_0 = M(-q)^{e(Y)}$ and it was determined by [Li06, BF08, LP09] separately. Here $M(q)$ denotes the *MacMahon function*, which is given by

$$M(q) = \prod_{k \geq 1} (1 - q^k)^{-k} = 1 + q + 3q^2 + 6q^3 + 13q^4 + 24q^5 + \dots \tag{3.2.6}$$

Note that $e(Y) = e(\mathbf{P}^1 \times \mathbf{P}^1) = 4$. This completes the computation on the resolution.

Fortunately, the Hilbert scheme of curves of class $(2,-2-f,4+f)$ on $\mathcal{X}$ is also smooth. To see this, we describe $\mathrm{H}_f := \mathrm{Hilb}_{\mathcal{X}}(2,-2-f,4+f)$ as a smooth Quot scheme, using the description of coherent sheaves on a total space as sheaves of algebras on the base.

Claim 3.2.3. Let $\mathrm{Q}_f(a) := \mathrm{Quot}_{\mathbf{P}^1}(\mathcal{O}^{\oplus 3}(a); \mathrm{rk} = 1, \deg = a+f)$ denote the Quot scheme parametrising rank one and degree $a+f$ quotients of the locally free sheaf $\mathcal{O}_{\mathbf{P}^1}^{\oplus 3}(a)$, where $f \geq 0$ and $a \in \mathbf{Z}$. There is an isomorphism $\alpha_f\colon \mathrm{Q}_f(2) \to \mathrm{H}_f$ of schemes.

The proof is a family version of the reasoning in Example 3.2.1.

Proof. Since both schemes are projective and of finite type, it suffices to consider families parametrised by affine schemes of finite type over $\mathbf{C}$. Thus let $\mathbf{C} \to \mathrm{R}$ be a finite type $\mathbf{C}$-algebra and consider a short exact sequence of coherent sheaves

$$0 \to \mathrm{K} \to \mathcal{O}_{\mathbf{P}^1_{\mathrm{R}}}(2)^{\oplus 3} \to \mathrm{Q} \to 0 \tag{3.2.7}$$

on $\mathbf{P}^1_{\mathrm{R}} := \mathbf{P}^1 \otimes_{\mathbf{C}} \mathrm{R}$ such that Q (and hence K) is R-flat and such that $\mathrm{rk}(\mathrm{Q}_r) = 1$ and $\deg(\mathrm{Q}_r) = 2+f$ for all closed points $r \in \mathrm{R}$. We construct an exact sequence

$$\begin{array}{ccccccccccccc}
\mathrm{I} & := & 0 & \oplus & 0 & \oplus & \mathrm{K}\otimes\rho^+ & \oplus & \mathcal{O}^-(3)^{\oplus 4} & \oplus & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathrm{B}\otimes\mathrm{R} & = & \mathcal{O}^+ & \oplus & \mathcal{O}^-(1)^{\oplus 2} & \oplus & \mathcal{O}^+(2)^{\oplus 3} & \oplus & \mathcal{O}^-(3)^{\oplus 4} & \oplus & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathrm{S} & := & \mathcal{O}^+ & \oplus & \mathcal{O}^-(1)^{\oplus 2} & \oplus & \mathrm{Q}\otimes\rho^+ & \oplus & 0 & \oplus & \dots
\end{array} \tag{3.2.8}$$

of $\mathbf{Z}_2$-equivariant sheaves of $\mathrm{B}\otimes\mathrm{R}$-modules on $\mathbf{P}^1_{\mathrm{R}}$. By construction, S is an R-flat sheaf and for every closed point $r \in \mathrm{R}$ we have $[\mathrm{S}_r] = (2,-2-f,4+f)$ as required. Let $\mathbf{Z}_2$ act trivially on $\mathrm{Spec}(\mathrm{R})$. Via the natural equivalences

$$\mathrm{Coh}^{\mathbf{Z}_2}(\mathbf{P}^1_{\mathrm{R}}, \mathrm{B}\otimes\mathrm{R}) \cong \mathrm{Coh}([\mathbf{P}^1/\mathbf{Z}_2]_{\mathrm{R}}, \mathrm{B}\otimes\mathrm{R}) \cong \mathrm{Coh}(\mathcal{X}_{\mathrm{R}}) \tag{3.2.9}$$

we have constructed a morphism $\alpha_f\colon \mathrm{Q}_f(2) \to \mathrm{H}_f$ of projective schemes, where $\mathrm{Coh}(\mathrm{X},\mathrm{F})$ denotes the category of coherent sheaves of F-modules on a noetherian X where F is a sheaf of $\mathcal{O}_{\mathrm{X}}$-algebras. Note that α_f is injective on R-valued points.

Conversely, an R-valued point of H_f induces a short exact sequence

$$0 \to \mathrm{I} \to \mathrm{B}\otimes\mathrm{R} \to \mathrm{S} \to 0 \tag{3.2.10}$$

of $\mathbf{Z}_2$-equivariant sheaves of $\mathrm{B}\otimes\mathrm{R}$-modules on $\mathbf{P}^1_{\mathrm{R}}$, where S (and hence I) is R-flat and $[\mathrm{S}_r] = (2,-2-f,4+f)$ for every closed point $r \in \mathrm{R}$. Note that $\mathrm{I} \subset \mathrm{B}\otimes\mathrm{R}$ is a subsheaf of

$B \otimes R$-algebras so it inherits a $\mathbf{Z}_{\geq 0}$-grading; similarly, S inherits a grading.

By the previous example, we have $B_{\geq 3} \subset I_r$ for every closed point $r \in R$, because each I_r corresponds to a quotient of $\mathcal{O}_{\mathcal{X}}$ on the extremal ray L.

We claim that the subset

$$G := \{r \in R \mid B_{\geq 3} \subset I_r\} \subset R \tag{3.2.11}$$

is open and, hence, that $G = R$. In other words, the property $B_{\geq 3} \subset I_r$ is open in flat families. To see this, note that I_r never contains B_0. Moreover, the condition that the $\mathbf{Z}_2$-anti-invariant part of S_r is $\mathcal{O}_C^-(1)^{\oplus 2}$ holds for all closed points. It is an open condition in flat families because the base curve $C \cong \mathbf{P}^1$ is rigid. The claim follows.

We conclude that I and S are fully determined by the exact sequence in degree 2

$$0 \to I \cap (B \otimes R)_2 \to (B \otimes R)_2 \to S_2 \to 0 \tag{3.2.12}$$

and I is as in diagram (3.2.8). Since I is R-flat, the same holds for S_2 and hence for S. We conclude that α_f is also surjective on R-valued points. This proves the claim. □

We note that tensoring by a line bundle induces an isomorphism of Quot schemes. In particular, $Q_f(a) \cong Q_f(b)$ for all $a, b \in \mathbf{Z}$. For the sake of notational simplicity, we study rank one quotients of $\mathcal{O}^{\oplus 3}$ to prove smoothness of these schemes.

Claim 3.2.4. The Quot scheme Q_f is smooth of dimension $3f + 2$ for all $f \geq 0$.

Proof. Smoothness is an open property and Q_f is of finite type, so it suffices to prove that Q_f is smooth at all of its closed points. Let $[q\colon \mathcal{O}^{\oplus 3} \twoheadrightarrow Q]$ be such a point, where $\mathrm{rk}(Q) = 1$ and $\deg(Q) = f$. The corresponding kernel $K = \ker(q)$ satisfies $\mathrm{rk}(K) = 2$ and $\deg(K) = -f$. Since $\mathbf{P}^1$ is a smooth projective curve and K is a torsion free sheaf, K is in fact a locally free sheaf.

By [HL10, Prop. 2.2.8], Q_f is smooth at $[q]$ of dimension $\hom(K, Q)$ provided that $\mathrm{Ext}^1(K, Q) = 0$. By Hirzebruch–Riemann–Roch, we find that

$$\chi(K, Q) = \int_{\mathbf{P}^1} (2, f) \cdot (1, f) \cdot (1, 1) = 3f + 2. \tag{3.2.13}$$

Moreover, K splits as $K = \mathcal{O}(a) \oplus \mathcal{O}(b)$ such that $a + b = -f$ and $a, b \leq 0$ since K injects into $\mathcal{O}^{\oplus 3}$. A simple cohomology computation shows that

$$\mathrm{Ext}^1(K, Q) = H^1(\mathbf{P}^1, Q(-a)) \oplus H^1(\mathbf{P}^1, Q(-b)) = 0 \tag{3.2.14}$$

because $\deg Q(-a), \deg Q(-b) \geq 0$ since $\mathcal{O}^{\oplus 3} \twoheadrightarrow Q$. This proves the claim. □

It remains to compute the topological Euler characteristic of Q_f for $f \geq 0$. There is a natural action of $(\mathbf{C}^\times)^3$ on $\mathcal{O}^{\oplus 3}$ scaling the fibres. It induces an action of $(\mathbf{C}^\times)^3$ on Q_f that is compatible with the induced torus action of $\mathbf{C}^\times$ on $\mathbf{P}^1$. The fixed points for this action are quotients of the form

$$q\colon \mathcal{O}^{\oplus 3} \twoheadrightarrow \mathcal{O} \oplus \mathrm{Z}, \tag{3.2.15}$$

where Z is a zero-dimensional sheaf of $\deg(\mathrm{Z}) = f$ such that $\mathrm{supp}(\mathrm{Z}) \subset \{0, \infty\} \subset \mathbf{P}^1$, and where q maps one of the three copies of $\mathcal{O}$ identically onto $\mathcal{O}$; there are three such choices. The other surjection $\mathcal{O}^{\oplus 2} \twoheadrightarrow \mathrm{Z}$ yields a partition $f_{0,1} + f_{\infty,1} + f_{0,2} + f_{\infty,2}$ of f with $f_{0,i}, f_{\infty,i} \geq 0$ for $i = 1, 2$; there are $\binom{f+3}{3}$ such partitions. Since $e(\mathbf{C}^\times) = 0$, it follows that

$$e(\mathrm{Q}_f) = 3\binom{f+3}{3} = \frac{1}{2}(f+1)(f+2)(f+3) \tag{3.2.16}$$

for $f \geq 0$. Hence $\mathrm{DT}_{\mathcal{X}}(2, -2-f, 4+f) = (-1)^f (f+1)(f+2)(f+3)/2$ for $f \geq 0$. Note that $(2, -2-f, 4+f) \notin \mathrm{NE}_{\mathrm{mr}}(\mathcal{X})$ for $f < 0$ forcing the invariants to vanish.

As for $\mathrm{DT}_0(\mathcal{X})$, the only contribution comes from classes $(0, -f, f)$ for $f \geq 0$. The only quotients of $\mathcal{O}_{\mathcal{X}}$ of class $(0, -1, 1)$ are stacky skyscrapers $\mathcal{O}_p^+$. Such a class can only be realised once per stacky point so $\mathrm{Hilb}_{\mathcal{X}}(0, -f, f) \cong \mathrm{Sym}^f(\mathbf{P}^1) \cong \mathbf{P}^f$. We conclude that $\mathrm{DT}_{\mathcal{X}}(0, -f, f) = (-1)^f (f+1)$ for $f \geq 0$ and zero otherwise. □

3.3 Rational functions and implications

Collecting the above invariants into their generating series, we obtain the following

Corollary 3.3.1. With $x = q^{(-1,1)}$, both sides of equation (3.0.2) yield rational functions

$$\left.\frac{\mathrm{DT}(\mathrm{Y})}{\mathrm{DT}(\mathrm{Y})_0}\right|_{\mathrm{L}} = 3q^{(0,2)}\left(\frac{1}{1+x}\right)^2 \overset{!}{=} 3q^{(-2,4)}\left(\frac{1}{1+x^{-1}}\right)^2 = \left.\frac{\mathrm{DT}_{\mathrm{mr}}(\mathcal{X})}{\mathrm{DT}_0(\mathcal{X})}\right|_{\mathrm{L}}$$

that are moreover equal as such after analytically continuing via $x \leftrightarrow x^{-1}$ at $\overset{!}{=}$.

Proof. We collect the above contributions in their generating function to find

$$\left.\frac{\mathrm{DT}(\mathrm{Y})}{\mathrm{DT}(\mathrm{Y})_0}\right|_{\mathrm{L}} = \sum_{f\in\mathbf{Z}} \mathrm{DT}_{\mathrm{Y}}(2, f, 2-f)\, q^{(f,2-f)} = 3q^{(0,2)}\left(\frac{1}{1+x}\right)^2. \tag{3.3.1}$$

The generating series on the orbifold are

$$\mathrm{DT}_{\mathrm{mr}}(\mathcal{X})|_{\mathrm{L}} = 3q^{(-2,4)}\left(\frac{1}{1+x^{-1}}\right)^4 \quad \text{and} \quad \mathrm{DT}_0(\mathcal{X})|_{\mathrm{M}} = \left(\frac{1}{1+x^{-1}}\right)^2 \tag{3.3.2}$$

Putting these series together completes the proof. □

Remark 3.3.2. In Theorem 4.3.18, we prove an orbifold DT/PT correspondence. Together with the correspondence of Bryan–Steinberg of Theorem 1.2.16, one may verify Corollary 3.3.1 by directly computing certain PT invariants on $\mathcal{X}$ and BS invariants on $f\colon \mathrm{Y} \to \mathrm{X}$.

We conclude that the crepant resolution conjecture is *not* true as an equality of generating series, but it *might* be true as an equality of rational functions. In the next two chapters, we show that this is indeed the case.

Remark 3.3.3. Note that our counterexample is computed in the geometry of a toric three-dimensional Calabi–Yau orbifold with transverse A_n-singularities in the sense of the work [Ros17] of D. Ross. His main result [Ros17, Thm. 2.2] claims a proof of the crepant resolution conjecture as an equality of generating series. This is in direct contradiction with our Corollary 3.3.1.

Tracing through the proof of [Ros17, Thm. 2.2], a counterexample to his key technical result [Ros17, Thm. 3.1] can be found[1] for the choice of 2-quotient $\lambda = (2, 2)$. This precisely corresponds to the class of two horizontal curves, the classes for which we have computed the Donaldson–Thomas invariants in Section 3.2.

[1] We thank Jørgen Rennemo for pointing this out.

Chapter 4

Pairs and their wall-crossing

We introduce the notion of a *pair*, a generalisation of the notion of a curve on a smooth three-dimensional Calabi–Yau orbifold $\mathcal{X}$. A pair is associated to a torsion pair on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$, which we think of as a rough notion of stability. Examples of pairs are ideal sheaves of curves and stable pairs in the sense of [PT10], thus putting these notions on an equal footing. We prove basic results about pairs, such as conditions which ensure that their moduli stack exists and is a $\mathbf{C}^*$-gerbe over its coarse moduli space. If this is the case, we obtain pair counting invariants by taking the Behrend weighted Euler characteristic of latter. Moreover, we establish a universal wall-crossing formula in a motivic Hall algebra relating all notions of pairs and hence, upon applying the integration map, the associated counting invariants.

As an application we prove the DT/PT correspondence for smooth CY3 orbifolds that satisfy the hard Lefschetz property and have a projective coarse moduli space.

Throughout, $\mathcal{X}$ will denote a projective CY3 orbifold as in Definition 2.1.21.

4.1 The categorical setting of pairs

Pairs are certain objects in an abelian category $\mathtt{A}$ satisfying $\mathrm{Coh}_{\leq 1}(\mathcal{X}) \subset \mathtt{A} \subset \mathrm{D}(\mathcal{X})$, introduced by Y. Toda in [Tod10a]. It contains ideal sheaves of curves, stable pairs, and pairs in our sense.

First, we introduce the category $\mathtt{A}$ and discuss some of its properties. Then we come to our definition of *pair* and provide a number of examples.

4.1.1 Toda's category $\mathtt{A}$

Following [Tod10a], the abelian category $\mathtt{A}$ is constructed as a subcategory of a category obtained as a tilt (see Definition 2.1.13) at a torsion pair in $\mathrm{Coh}(\mathcal{X})$.

The construction goes as follows. Consider the full subcategory $\mathrm{Coh}_{\leq 1}(\mathcal{X}) \subset \mathrm{Coh}(\mathcal{X})$. Its right orthogonal subcategory is $\mathrm{Coh}_{\geq 2}(\mathcal{X})$, i.e., the full subcategory of sheaves that admit no subsheaves of dimension at most one.

Lemma 4.1.1. The pair $(\mathrm{Coh}_{\leq 1}(\mathcal{X}), \mathrm{Coh}_{\geq 2}(\mathcal{X}))$ define a torsion pair on $\mathrm{Coh}(\mathcal{X})$.

Proof. By Lemma 2.1.16, $\mathrm{Coh}(\mathcal{X})$ is a noetherian abelian category. Clearly, the subcategory $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ is closed under extensions and quotients in $\mathrm{Coh}(\mathcal{X})$. The conditions of Lemma 2.1.17 are met, so we may conclude. □

Definition 4.1.2. Tilting $\mathrm{Coh}(\mathcal{X})$ at the above torsion pair yields the heart

$$\mathrm{Coh}^\flat(\mathcal{X}) := \langle \mathrm{Coh}_{\geq 2}(\mathcal{X})[1], \mathrm{Coh}_{\leq 1}(\mathcal{X})\rangle \subset \mathrm{D}^{[-1,0]}(\mathcal{X}). \tag{4.1.1}$$

of a bounded t-structure on $\mathrm{D}(\mathcal{X})$.

Remark 4.1.3. Even though $\mathrm{Coh}(\mathcal{X})$ is a noetherian category, its tilt $\mathrm{Coh}^\flat(\mathcal{X})$ is not. Indeed, let $\mathrm{C} \subset \mathrm{D} \subset \mathcal{X}$ be the inclusion of a curve in a divisor on $\mathcal{X}$. By rotating the short exact sequence $0 \to \mathcal{O}_\mathrm{D} \to \mathcal{O}_\mathrm{D}(\mathrm{C}) \to \mathcal{O}_\mathrm{C}(\mathrm{C}) \to 0$, we obtain an exact triangle

$$\mathcal{O}_\mathrm{C}(\mathrm{C}) \to \mathcal{O}_\mathrm{D}[1] \xrightarrow{\phi} \mathcal{O}_\mathrm{D}(\mathrm{C})[1] \tag{4.1.2}$$

in $\mathrm{D}(\mathcal{X})$. By Remark 2.1.19 this is a short exact sequence in $\mathrm{Coh}^\flat(\mathcal{X})$ since all its vertices lie in $\mathrm{Coh}^\flat(\mathcal{X})$. It follows that ϕ is surjective in $\mathrm{Coh}^\flat(\mathcal{X})$. Twisting by $\mathcal{O}_\mathrm{D}(\mathrm{C})$ induces a non-trivial chain of surjections in $\mathrm{Coh}^\flat(\mathcal{X})$

$$\mathcal{O}_\mathrm{D}[1] \twoheadrightarrow \mathcal{O}_\mathrm{D}(\mathrm{C})[1] \twoheadrightarrow \mathcal{O}_\mathrm{D}(2\mathrm{C})[1] \twoheadrightarrow \dots \tag{4.1.3}$$

that does not stabilise. Hence $\mathrm{Coh}^\flat(\mathcal{X})$ is not noetherian.

As will be shown in Lemma 4.2.6, the moduli of objects in $\mathrm{Coh}^\flat(\mathcal{X})$ are well-behaved. However, in order to construct torsion pairs using Lemma 2.1.17, we need to consider a suitable *noetherian* subcategory $\mathtt{A} \subset \mathrm{Coh}^\flat(\mathcal{X})$ following [Tod10a].

The category $\mathtt{A}$ is constructed as a subcategory of $\mathrm{Coh}^\flat(\mathcal{X})$ by restricting the bounded t-structure it induces on $\mathrm{D}(\mathcal{X})$ to a certain triangulated subcategory.

Definition 4.1.4. The category $\mathtt{A}$ is the full subcategory of $\mathrm{Coh}^\flat(\mathcal{X})$ defined as

$$\mathtt{A} = \langle \mathcal{O}_\mathcal{X}[1], \mathrm{Coh}_{\leq 1}(\mathcal{X})\rangle_{\mathrm{ex}} \subset \mathrm{Coh}^\flat(\mathcal{X}) \subset \mathrm{D}(\mathcal{X}), \tag{4.1.4}$$

where ex denotes extension-closure in the abelian category $\mathrm{Coh}^\flat(\mathcal{X})$.

We collect all the basic facts we need about $\mathtt{A}$ in the following lemma.

Lemma 4.1.5. The category $\mathtt{A}$ satisfies the following properties.

1. $\mathtt{A}$ is a noetherian abelian category with exact inclusion $\mathtt{A} \subset \mathrm{D}(\mathcal{X})$.
2. A short exact sequence in $\mathtt{A}$ is an exact triangle $\mathrm{A} \to \mathrm{B} \to \mathrm{C} \to \mathrm{A}[1]$ in $\mathrm{D}(\mathcal{X})$ such that $\mathrm{A}, \mathrm{B}, \mathrm{C} \in \mathtt{A}$.
3. If $\mathrm{E} \in \mathtt{A}$, then E is quasi-isomorphic to a two-term complex such that $\mathrm{H}^{-1}(\mathrm{E})$ is torsion free (or zero), and $\mathrm{H}^0(\mathrm{E}) \in \mathrm{Coh}_{\le 1}(\mathcal{X})$.
4. The full subcategory $\mathrm{Coh}_{\le 1}(\mathcal{X}) \subset \mathtt{A}$ is closed under extensions, quotients, and subobjects, and the inclusion $\mathrm{Coh}_{\le 1}(\mathcal{X}) \subset \mathtt{A}$ is exact.
5. $\mathtt{A}$ contains the shifted structure sheaf $\mathcal{O}_{\mathcal{X}}[1]$, the shifted ideal sheaf $\mathrm{I}_{\mathrm{C}}[1]$ of any curve $\mathrm{C} \subset \mathcal{X}$, and stable pairs.

Proof. The first item is proven in [Tod10a, Lem. 3.5, 3.8], and the second follows from Remark 2.1.19. The reader can readily verify the claims in the third and fourth items.

As for the fifth and final one, let $\mathrm{C} \subset \mathcal{X}$ be a curve. The ideal sheaf exact sequence $0 \to \mathrm{I}_{\mathrm{C}} \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathrm{C}} \to 0$ in $\mathrm{Coh}(\mathcal{X})$ defines an exact triangle in $\mathrm{D}(\mathcal{X})$ that rotates to a triangle

$$\mathcal{O}_{\mathrm{C}} \to \mathrm{I}_{\mathrm{C}}[1] \to \mathcal{O}_{\mathcal{X}}[1]. \tag{4.1.5}$$

By extension-closure, we find that $\mathrm{I}_{\mathrm{C}}[1] \in \mathtt{A}$ because $\mathcal{O}_{\mathrm{C}}, \mathcal{O}_{\mathcal{X}}[1] \in \mathtt{A}$. In particular, the above triangle defines an exact sequence in $\mathtt{A}$.

Similarly, recall that a stable pair in the sense of [PT09] is a two-term complex

$$\mathrm{E} = (\mathcal{O}_{\mathcal{X}} \xrightarrow{s} \mathrm{F}) \in \mathrm{D}^{[-1,0]}(\mathcal{X}) \tag{4.1.6}$$

where $\mathrm{F} \in \mathrm{Coh}_{\le 1}(\mathcal{X})$ is pure and $\mathrm{coker}(s) \in \mathrm{Coh}_0(\mathcal{X})$. Such a complex fits into an exact triangle $\mathrm{F} \to \mathrm{E} \to \mathcal{O}_{\mathcal{X}}[1]$ in $\mathrm{D}(\mathcal{X})$, so $\mathtt{A}$ contains stable pairs by the same argument. □

Remark 4.1.6. Let $\mathrm{N}(\mathtt{A}) \subset \mathrm{N}(\mathcal{X})$ denote the subgroup generated by objects in $\mathtt{A}$. By part (3) of Lemma 4.1.5, the inclusion $\mathrm{Coh}_{\le 1}(\mathcal{X}) \subset \mathtt{A}$ induces an injection of abelian groups $i\colon \mathrm{N}_{\le 1}(\mathcal{X}) \hookrightarrow \mathrm{N}(\mathtt{A})$; the image of a class $\alpha \in \mathrm{N}(\mathtt{A})$ in the cokernel of i equals $\mathrm{rk}(\alpha)$. We fix a splitting of i, so $\mathrm{N}(\mathtt{A}) = \mathbf{Z} \oplus \mathrm{N}_{\le 1}(\mathcal{X})$, that satisfies $i(\alpha) = (0, \alpha)$ for $\alpha \in \mathrm{N}_{\le 1}(\mathcal{X})$.

Finally, it is important that $\mathtt{A}$ contains Bryan–Steinberg pairs as well.

Lemma 4.1.7. Assume that $\mathcal{X}$ satisfies the hard Lefschetz condition and let $f\colon \mathrm{Y} \to \mathrm{X}$ denote the natural crepant resolution. The McKay equivalence $\Phi\colon \mathrm{D}(\mathrm{Y}) \to \mathrm{D}(\mathcal{X})$ of Theorem 2.4.11 sends Bryan–Steinberg pairs into $\mathtt{A} \subset \mathrm{D}(\mathcal{X})$.

Proof. Let $\mathrm{E} = (s\colon \mathcal{O}_{\mathrm{Y}} \to \mathrm{F}) \in \mathrm{D}^{[-1,0]}(\mathrm{Y})$ be a Bryan–Steinberg pair relative to f, i.e., F is a one-dimensional sheaf on Y such that $\mathrm{Hom}(\mathrm{T}_f, \mathrm{F}) = 0$ and the cokernel of s lies in

$$\mathrm{T}_f = \{\mathrm{T} \in \mathrm{Coh}_{\leq 1}(\mathrm{Y}) \mid \mathbf{R}f_*(\mathrm{T}) \in \mathrm{Coh}_0(\mathrm{X})\}. \tag{4.1.7}$$

The pair E fits into an exact triangle $\mathcal{O}_{\mathrm{Y}} \xrightarrow{s} \mathrm{F} \to \mathrm{E} \to \mathcal{O}_{\mathrm{Y}}[1]$ in $\mathrm{D}(\mathrm{Y})$. The McKay equivalence sends this to an exact triangle

$$\mathcal{O}_{\mathcal{X}} \xrightarrow{t} \Phi(\mathrm{F}) \to \Phi(\mathrm{E}) \to \mathcal{O}_{\mathcal{X}}[1] \tag{4.1.8}$$

in $\mathrm{D}(\mathcal{X})$ where $t = \Phi(s)$. By extension-closure of $\mathtt{A}$, it suffices to show $\Phi(\mathrm{F}) \in \mathrm{Coh}_{\leq 1}(\mathcal{X})$.

To see this, recall J. Calabrese's equivalence of categories $\Phi\colon \mathrm{Per}(\mathrm{Y}/\mathrm{X}) \to \mathrm{Coh}(\mathcal{X})$ from Theorem 2.4.20, where $\mathrm{Per}(\mathrm{Y}/\mathrm{X})$ is T. Bridgeland's category of 0-perverse sheaves. It contains the full subcategory $\mathtt{T} \subset \mathrm{Per}(\mathrm{Y}/\mathrm{X})$, the torsion part of the perverse torsion pair, where

$$\mathtt{T} := {}^0\mathtt{T} = \{\mathrm{T} \in \mathrm{Coh}(\mathrm{Y}) \mid \mathbf{R}^1 f_*(\mathrm{T}) = 0\}. \tag{4.1.9}$$

We claim that $\mathrm{F} \in \mathtt{T}$, whence it follows that $\Phi(\mathrm{F}) \in \mathrm{Coh}_{\leq 1}(\mathcal{X})$, completing the proof. Indeed, taking the image of the section s induces a short exact sequence

$$0 \to \mathrm{im}(s) \to \mathrm{F} \to \mathrm{coker}(s) \to 0 \tag{4.1.10}$$

in $\mathrm{Coh}_{\leq 1}(\mathrm{Y})$. Now $\mathcal{O}_{\mathrm{Y}} \in \mathtt{T}$ since X has rational singularities. We deduce that $\mathrm{im}(s) \in \mathtt{T}$ since $\mathtt{T}$ is closed under quotients. Clearly $\mathtt{T}_f \subset \mathtt{T}$, so $\mathrm{coker}(s)$ is an object in $\mathtt{T}$ as well. But $\mathtt{T}$ is closed under extension, so the claim follows. □

The rank of an object $\mathrm{E} \in \mathrm{Coh}^\flat(\mathcal{X})$ is non-positive since $\mathrm{rk}(\mathrm{E}) = -\,\mathrm{rk}\,\mathrm{H}^{-1}(\mathrm{E})$. From the above examples it is clear that we are particularly interested in objects of $\mathtt{A}$ of 'small' rank. There is an easy criterion to recognize when such objects in $\mathrm{Coh}^\flat(\mathcal{X})$ lie in $\mathtt{A}$.

Proposition 4.1.8. Let $\mathrm{E} \in \mathrm{Coh}^\flat(\mathcal{X})$ be an object.

1. If $\mathrm{rk}(\mathrm{E}) = 0$, then $\mathrm{E} \in \mathtt{A}$ if and only if $\mathrm{H}^{-1}(\mathrm{E}) = 0$;
2. If $\mathrm{rk}(\mathrm{E}) = -1$, then $\mathrm{E} \in \mathtt{A}$ if and only if $\mathrm{H}^{-1}(\mathrm{E})$ is torsion free and $\det(\mathrm{E}) = \mathcal{O}_{\mathcal{X}}$.

Importantly, these characterisations are well-behaved in flat families; see Lemma 4.2.7. We denote the subcategory of objects in $\mathtt{A}$ of small rank (equal to $0, -1$) by $\mathtt{S} \subset \mathtt{A}$. Moreover, we remark that the following proof does not use the condition $\mathrm{H}^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = 0$.

Proof. The first claim is immediate as any rank zero object in $\mathrm{Coh}^\flat(\mathcal{X})$ lies in the subcategory $\langle \mathrm{Coh}_2(\mathcal{X})[1], \mathrm{Coh}_{\leq 1}(\mathcal{X})\rangle$. As for the second, let $\mathrm{E} \in \mathrm{Coh}^\flat(\mathcal{X})$ be an object of $\mathrm{rk}(E) = -1$ and assume that $H^{-1}(E)$ is torsion free and $\det(E) = \mathcal{O}_{\mathcal{X}}$. By additivity of the determinant and $\operatorname{codim} H^0(E) \geq 2$, it follows from the short exact sequence

$$0 \to H^{-1}(E)[1] \to E \to H^0(E) \to 0$$

in $\mathrm{Coh}^\flat(\mathcal{X})$ that $\det H^{-1}(E) = \mathcal{O}_{\mathcal{X}}$. We write $I := H^{-1}(E)$. By extension-closure of $\mathtt{A}$, combined with the fact that $H^0(E) \in \mathrm{Coh}_{\leq 1}(\mathcal{X}) \subset \mathtt{A}$, it suffices to show that $I[1] \in \mathtt{A}$.

We claim that I is the ideal sheaf of a curve. To see this, let $a\colon A \twoheadrightarrow \mathcal{X}$ be an étale atlas of $\mathcal{X}$, i.e., A is a smooth threefold and a is an étale surjection; such an atlas exists because $\mathcal{X}$ is a smooth Deligne–Mumford stack. Let $I^* := \underline{\mathrm{Hom}}(I, \mathcal{O}_{\mathcal{X}})$ denote the dual sheaf of I. Since I is torsion free, it embeds into its double dual $i\colon I \hookrightarrow I^{**}$ via the canonical morphism.

We claim that $I^{**} = \mathcal{O}_{\mathcal{X}}$. Indeed, pulling back by an étale morphism commutes with taking the dual, so $a^*(I)$ is a rank one torsion free sheaf with trivial determinant on the smooth threefold A. Moreover, it has reflexive hull $a^*(I^{**})$. The latter sheaf is of rank one, hence locally free, a property which descends over a to I^{**}. But $\mathrm{rk}\, I^{**} = 1$ and $\det(I^{**}) = \mathcal{O}_{\mathcal{X}}$, so we find $I^{**} = \mathcal{O}_{\mathcal{X}}$. Moreover, by similarly pulling $\mathrm{coker}(i)$ back to A and using the right exactness of a^*, we deduce that $\dim \mathrm{coker}(i) \leq 1$. We conclude that I is the ideal sheaf of a curve, hence $I[1] \in \mathtt{A}$ by Lemma 4.1.5.

The converse is immediate. □

Corollary 4.1.9. Let $E \in \mathtt{A}$ be an object of class $(-1, \beta, c) \in N(\mathtt{A})$. Then

1. $H^{-1}(E)$ is the ideal sheaf of a curve $C \subset \mathcal{X}$ of class $\beta \geq \beta_C$, and
2. we have $\beta \geq \beta_T$, where $T = H^0(E)$, and $\beta \geq 0$.

Proof. There is an exact sequence $0 \to I_C[1] \to E \to T \to 0$ in $\mathtt{A}$ where $C \subset \mathcal{X}$ is a curve, and $T \in \mathrm{Coh}_{\leq 1}(\mathcal{X})$. Thus $\beta = \beta_C + \beta_T \geq 0$ where $[\mathcal{O}_C] = (\beta_C, c_C)$ and $[T] = (\beta_T, c_T)$. □

4.1.2 Pairs and examples

We motivate the definition of *pairs* via the example of PT or stable pairs.

Example 4.1.10. Recall that a PT pair on $\mathcal{X}$ is a pair (G, s) where G is a pure one-dimensional sheaf and $s\colon \mathcal{O}_{\mathcal{X}} \to G$ is a section such that $\dim \mathrm{coker}(s) \leq 0$. By [PT10, §2.2], the moduli space of such objects is isomorphic to the moduli space of two-term complexes $E = (s\colon \mathcal{O}_{\mathcal{X}} \to G)$ such that G and s satisfy the above properties. Note that by Lemma 4.1.5, $E \in \mathtt{A}$ and $\mathrm{rk}(E) = -1$.

Set $\mathtt{T}_{PT} = \mathrm{Coh}_0(\mathcal{X})$. We may restate the fact that $E \in \mathtt{A}$ is a PT pair as follows.

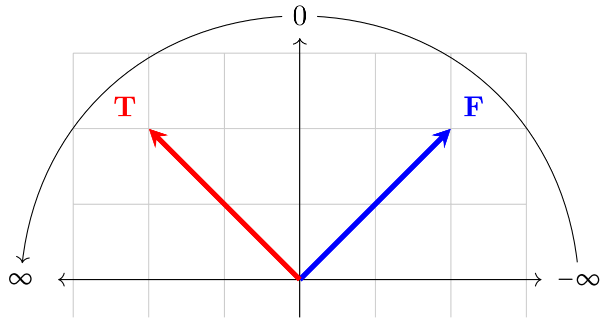
1. The sheaf G is pure if and only if $\mathrm{Hom}(T, G) = 0$ for all $T \in \mathtt{T}_{\mathrm{PT}}$. Recall the short exact sequence $0 \to G \to E \to \mathcal{O}_{\mathcal{X}}[1] \to 0$ in $\mathtt{A}$. By Serre duality it follows that $\mathrm{Hom}(T, \mathcal{O}_{\mathcal{X}}[1]) = H^2(\mathcal{X}, T) = 0$. Applying the functor $\mathrm{Hom}(T, -)$ to the exact sequence shows that G is pure if and only $\mathrm{Hom}(T, E) = 0$ for all $T \in \mathtt{T}_{\mathrm{PT}}$.

2. Recall the cohomology exact sequence $0 \to H^{-1}(E)[1] \to E \to H^0(E) \to 0$ in $\mathtt{A}$, where $H^0(E) = \mathrm{coker}(s)$. Set $\mathtt{F}_{\mathrm{PT}} = \mathrm{Coh}_1(\mathcal{X})$. Now it follows that $H^0(E) \in \mathtt{T}_{\mathrm{PT}}$ if and only if $\mathrm{Hom}(H^0(E), F) = 0$ for all $F \in \mathtt{F}_{\mathrm{PT}}$. Applying $\mathrm{Hom}(-, F)$ to the above sequence shows that $\mathrm{coker}(s) \in \mathtt{T}_{\mathrm{PT}}$ if and only if $\mathrm{Hom}(E, F) = 0$ for all $F \in \mathtt{F}_{\mathrm{PT}}$.

We have arrived at a *stability-type* characterisation of a stable pair $E \in \mathtt{A}$, using the fact that $(\mathrm{Coh}_0(\mathcal{X}), \mathrm{Coh}_1(\mathcal{X}))$ defines a torsion pair on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$. Indeed, we have found that a stable pair is a complex $E = (s\colon \mathcal{O}_{\mathcal{X}} \to G)$ in $\mathtt{A}$ such that $\mathrm{rk}(E) = -1$ and

$$\mathrm{Hom}(\mathtt{T}_{\mathrm{PT}}, E) = 0 = \mathrm{Hom}(E, \mathtt{F}_{\mathrm{PT}}). \tag{4.1.11}$$

In other words, we think of objects in $\mathtt{T}_{\mathrm{PT}}$ as having *larger slope* than E, of objects in $\mathtt{F}_{\mathrm{PT}}$ as having *smaller slope* than E, and of E as being 'stable'. In a picture:



(4.1.12)

The values on the circle encode the 'slopes' of objects with respect to a stability condition σ on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$. We plot the class of an object $G \in \mathrm{Coh}_{\leq 1}(\mathcal{X})$ on the ray through the origin and $\sigma(G)$. We picture pairs as lying on the vertical axis with slope zero. However, we emphasize that our 'stability condition' is only defined for objects in $\mathrm{Coh}_{\leq 1}(\mathcal{X})$.

Remark 4.1.11. Most of the torsion pairs we consider in the proof of the crepant resolution conjecture arise from an actual stability condition as follows. Given a stability condition on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ as in Definition 2.1.23, we may collapse the Harder–Narasimhan filtration of σ into a torsion pair

$$\begin{aligned} \mathtt{T}_\sigma &:= \langle T \in \mathrm{Coh}_{\leq 1}(\mathcal{X}) \mid T \text{ is } \sigma\text{-semistable, } \sigma(T) \geq 0 \rangle_{\mathrm{ex}} \\ \mathtt{F}_\sigma &:= \langle F \in \mathrm{Coh}_{\leq 1}(\mathcal{X}) \mid F \text{ is } \sigma\text{-semistable, } \sigma(F) < 0 \rangle_{\mathrm{ex}} \end{aligned} \tag{4.1.13}$$

The above picture then captures the notion of a $(\mathtt{T}_\sigma, \mathtt{F}_\sigma)$-pair.

Remark 4.1.12. It will follow from the criterion of Proposition 4.1.16 that any object $E \in \mathtt{A}$ of rank -1 satisfying the properties of equation (4.1.11) is a PT pair, i.e., E is of the form $(s\colon \mathcal{O}_{\mathcal{X}} \to G)$ with $G \in \mathtt{F}_{PT}$ and $\mathrm{coker}(s) \in \mathtt{T}_{PT}$.

We emphasize that not every pair is of this *standard* form; see Remark 4.1.19.

Motivated by this example we now come to the definition of a pair. For later convenience we state it in a general context, where $(\mathtt{T}, \mathtt{F})$ need not be a torsion pair but the property $\mathrm{Hom}(T, F) = 0$ for all $T \in \mathtt{T}$ and $F \in \mathtt{F}$ does hold. The picture we have in mind is that there exists an intermediate subcategory $\mathtt{W} \subset \mathrm{Coh}_{\leq 1}(\mathcal{X})$ of *walls* containing objects with lower slope than those in $\mathtt{T}$, and with larger slope than those in $\mathtt{F}$.

Remark 4.1.13. This idea will be formalised shortly in the notion of a *torsion triple*, which we denote by $\langle \mathtt{T}, \mathtt{W}, \mathtt{F} \rangle$; see Definition 4.1.22.

Definition 4.1.14. Let $(\mathtt{T}, \mathtt{F})$ be a pair of additive subcategories of $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ such that $\mathrm{Hom}(T, F) = 0$ for all $T \in \mathtt{T}$ and $F \in \mathtt{F}$. A $(\mathtt{T}, \mathtt{F})$*-pair* is an object $E \in \mathtt{A}$ of $\mathrm{rk}(E) = -1$ such that

1. $\mathrm{Hom}(T, E) = 0$ for all $T \in \mathtt{T}$,

2. $\mathrm{Hom}(E, F) = 0$ for all $F \in \mathtt{F}$.

When no confusion is likely to arise, we refer to such objects simply as *pairs*. We write $\mathtt{Pair}(\mathtt{T}, \mathtt{F}) \subset \mathtt{A}$ for the corresponding subcategory.

Remark 4.1.15. Assuming $(\mathtt{T}, \mathtt{F})$ is a *torsion* pair, two things follow. The third condition is equivalent to $H^0(E) \in \mathtt{T}$. Moreover, if E is a pair of the form $\mathcal{O}_{\mathcal{X}} \to G$ for some $G \in \mathrm{Coh}_{\leq 1}(\mathcal{X})$, then the second condition is equivalent to $G \in \mathtt{F}$.

Under a cohomological criterion on $\mathtt{T}$, all pairs are of a standard form.

Proposition 4.1.16. Let $(\mathtt{T}, \mathtt{F})$ be a torsion pair on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ such that every $T \in \mathtt{T}$ satisfies $H^i(\mathcal{X}, T) = 0$ for all $i \neq 0$. Then an object $E \in \mathtt{A}$ of rank -1 is a $(\mathtt{T}, \mathtt{F})$-pair if and only if it is quasi-isomorphic to a two-term complex

$$E = (\mathcal{O}_{\mathcal{X}} \xrightarrow{s} G)$$

with $H^0(E) = \mathrm{coker}(s) \in \mathtt{T}$ and $G \in \mathtt{F}$.

Proof. The proof of [Tod10a, Lem. 3.11(ii)] goes through verbatim. ☐

Example 4.1.17. As was observed in Remark 4.1.12, by the cohomological criterion a $(\mathtt{T}_{PT}, \mathtt{F}_{PT})$-pair is the same thing as a PT stable pair in the sense of [PT09].

Example 4.1.18. Consider the trivial torsion pair $\mathtt{T}_{\mathrm{DT}} = 0$ and $\mathtt{F}_{\mathrm{DT}} = \mathrm{Coh}_{\leq 1}(\mathcal{X})$. The cohomological criterion applies, so any $(\mathtt{T}_{\mathrm{DT}}, \mathtt{F}_{\mathrm{DT}})$-pair E is of the form

$$\mathrm{E} = (s\colon \mathcal{O}_{\mathcal{X}} \to \mathrm{G}) \in \mathtt{A} \subset \mathrm{D}^{[-1,0]}(\mathcal{X}) \tag{4.1.14}$$

such that $\mathrm{H}^0(\mathrm{E}) = \mathrm{coker}(s) \in \mathtt{T}_{\mathrm{DT}} = 0$. In other words, s is a surjection in $\mathrm{Coh}(\mathcal{X})$ and $\mathrm{G} = \mathcal{O}_{\mathrm{C}}$ for a closed subscheme $\mathrm{C} \subset \mathcal{X}$, and $\mathrm{E} = \mathrm{I}_{\mathrm{C}}[1]$ is the ideal sheaf of a curve by Corollary 4.1.9. This explains the notation $(\mathtt{T}_{\mathrm{DT}}, \mathtt{F}_{\mathrm{DT}})$ for such pairs.

Remark 4.1.19. There are many torsion pairs for which the assertion of Proposition 4.1.16 fails to hold. For example, it fails in the case of the perverse torsion pair on $\mathrm{Coh}(\mathrm{Y})$ whose tilt $\mathrm{Per}(\mathrm{Y}/\mathrm{X})$ is identified with $\mathrm{Coh}(\mathcal{X})$ by the McKay equivalence Φ.

This failure of the criterion is the main complication in proving the crepant resolution conjecture for Donaldson–Thomas invariants, and it is the fundamental obstruction preventing it from being true as an equality of generating series.

As a concrete, albeit contrived, example take $\mathtt{T} = \mathrm{Coh}_{\leq 1}(\mathcal{X})$. A $(\mathtt{T}, 0)$-pair E fits into an exact sequence $0 \to \mathrm{I}_{\mathrm{C}}[1] \to \mathrm{E} \to \mathrm{T} \to 0$ for $\mathrm{C} \subset \mathcal{X}$ a curve and $\mathrm{T} \in \mathtt{T}$ by Corollary 4.1.9. However, since $\mathrm{Hom}(\mathtt{T}, \mathrm{E}) = 0$ we find

$$\mathrm{Hom}(\mathcal{O}_{\mathrm{C}}, \mathrm{E}) = \mathrm{Hom}(\mathcal{O}_{\mathrm{C}}, \mathrm{I}_{\mathrm{C}}[1]) = \mathrm{Ext}^1(\mathcal{O}_{\mathrm{C}}, \mathrm{I}_{\mathrm{C}}) = 0 \tag{4.1.15}$$

contradicting the fact that the exact sequence $\mathrm{I}_{\mathrm{C}} \hookrightarrow \mathcal{O}_{\mathrm{Y}} \twoheadrightarrow \mathcal{O}_{\mathrm{C}}$ is not split.

Finally, we explain where Bryan–Steinberg pairs fit in.

Example 4.1.20. The construction of the category $\mathtt{A}$ goes through on any smooth projective Calabi–Yau threefold Y; in fact, Y. Toda originally introduced this category for such varieties in [Tod10a, §3].

To avoid confusion, we keep track of the variety in the notation of $\mathtt{A}$, and write

$$\mathtt{A}_{\mathrm{Y}} \coloneqq \langle \mathcal{O}_{\mathrm{Y}}[1], \mathrm{Coh}_{\leq 1}(\mathrm{Y}) \rangle_{\mathrm{ex}} \subset \mathrm{D}^{[-1,0]}(\mathrm{Y}). \tag{4.1.16}$$

Recall the Bryan–Steinberg torsion pair induced by the full subcategory

$$\mathtt{T}_f \coloneqq \{\mathrm{T} \in \mathrm{Coh}_{\leq 1}(\mathrm{Y}) \mid \dim(f_*\mathrm{T}) \leq 0,\, \mathbf{R}^1 f_*(\mathrm{T}) = 0\} \tag{4.1.17}$$

and let $\mathtt{F}_f \coloneqq \mathtt{T}_f^{\perp}$ denote its (right) complement. Also recall that $(\mathtt{T}_f, \mathtt{F}_f)$ defines a torsion pair on $\mathrm{Coh}_{\leq 1}(\mathrm{Y})$ by Lemma 2.4.27.

Let $\mathrm{T} \in \mathtt{T}_f$, so in particular $\mathbf{R}f_*(\mathrm{T}) = f_*(\mathrm{T})$ is a zero-dimensional sheaf on X. A direct application of the Leray spectral sequence $\mathrm{H}^i(\mathrm{X}, \mathbf{R}^j f_*(\mathrm{T})) \Rightarrow \mathrm{H}^{i+j}(\mathrm{Y}, \mathrm{T})$ shows that $H^i(Y,T) = 0$ for all $i > 0$. The cohomological criterion of Proposition 4.1.16 applies and we deduce that a $(\mathtt{T}_f, \mathtt{F}_f)$-pair $E \in \mathtt{A}_Y$ is the same as a Bryan–Steinberg pair.

Remark 4.1.21. As these example show, the notion of $(\mathtt{T},\mathtt{F})$-pair is a generalisation in the spirit of Y. Toda's approach to curve counting. Indeed, they are two-term complexes in the bounded derived category, as opposed to sheaves equipped with a section as in T. Bridgeland's, J. Calabrese's, and J. Bryan and D. Steinberg's approaches in proving their respective comparison theorems [Bri11, Cal16a, Cal16b, BS16].

We now explain why the category $\mathtt{A}$ and the notion of a $(\mathtt{T},\mathtt{F})$-pair interact well. Recall the following natural generalisation of the notion of torsion pair from [Tod16a].

Definition 4.1.22. Let $(\mathtt{B}_1, \mathtt{B}_2, \dots, \mathtt{B}_n)$ be a tuple of full additive subcategories of an abelian category $\mathtt{B}$. These form a *torsion n-tuple*, notation $\mathtt{B} = \langle \mathtt{B}_1, \mathtt{B}_2, \dots, \mathtt{B}_n \rangle$, if

1. there are no maps $\mathrm{Hom}(\mathtt{B}_i, \mathtt{B}_j) = 0$ for $i < j$, and
2. for every object $E \in \mathtt{B}$ there is a filtration

$$0 = E_0 \subset E_1 \subset \dots \subset E_n = E$$

in $\mathtt{B}$ such that $F_i = E_i / E_{i-1} \in \mathtt{B}_i$ for all $i = 1, 2, \dots, n$.

The first condition implies that the filtration in the second condition is unique.

Remark 4.1.23. Note that for any $1 \leq i \leq n-1$ we obtain a torsion pair

$$\mathtt{B} = \langle \langle \mathtt{B}_1, \dots, \mathtt{B}_i \rangle, \langle \mathtt{B}_{i+1} \dots, \mathtt{B}_n \rangle \rangle$$

by collapsing the filtration into $0 \to E_i \to E \to E/E_i \to 0$ in $\mathtt{B}$.

A key feature of the category $\mathtt{A}$ is that any torsion pair on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ induces a torsion triple on $\mathtt{A}$. This means we can control the notion of $(\mathtt{T},\mathtt{F})$-pair in $\mathtt{A}$ by working with the corresponding torsion pair on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$.

Indeed, let $(\mathtt{T},\mathtt{F})$ be a torsion pair on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$. Define the full subcategory

$$\mathtt{V}(\mathtt{T},\mathtt{F}) = \{E \in \mathtt{A} \mid \mathrm{Hom}(\mathtt{T}, E) = 0 = \mathrm{Hom}(E, \mathtt{F})\} \subset \mathtt{A}. \tag{4.1.18}$$

Clearly, this category contains all $(\mathtt{T},\mathtt{F})$-pairs.

Proposition 4.1.24. Let $(\mathtt{T},\mathtt{F})$ be a torsion pair on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$. There is an induced torsion triple

$$\mathtt{A} = \langle \mathtt{T}, \mathtt{V}(\mathtt{T},\mathtt{F}), \mathtt{F} \rangle \tag{4.1.19}$$

on the noetherian abelian category $\mathtt{A}$.

Proof. To construct the induced torsion triple, we use Lemma 2.1.17 to define two torsion pairs on $\mathtt{A}$. During the construction, we repeatedly use the facts of Lemma 4.1.5 that the subcategory $\mathrm{Coh}_{\leq 1}(\mathcal{X}) \subset \mathtt{A}$ is closed under extensions and quotients.

For the first torsion pair, note that the subcategory $\mathtt{T} \subset \mathrm{Coh}_{\leq 1}(\mathcal{X}) \subset \mathtt{A}$ is also closed under extensions and quotients, and that the abelian category $\mathtt{A}$ is noetherian. By Lemma 2.1.17, we obtain a torsion pair $\mathtt{A} = \langle \mathtt{T}, \mathtt{G} \rangle$ where

$$\mathtt{G} = \mathtt{T}^{\perp} \equiv \{\mathrm{G} \in \mathtt{A} \mid \mathrm{Hom}(\mathrm{T}, \mathrm{G}) = 0 \text{ for all } \mathrm{T} \in \mathtt{T}\}. \tag{4.1.20}$$

For the second torsion pair, consider the full subcategory

$$\mathtt{T}' := \{\mathrm{E} \in \mathtt{A} \mid \mathrm{H}^0(\mathrm{E}) \in \mathtt{T}\}. \tag{4.1.21}$$

It is closed under extensions and quotients. By Lemma 2.1.17, we obtain a torsion pair $\mathtt{A} = \langle \mathtt{T}', \mathtt{F}' \rangle$. The decomposition of an object $\mathrm{E} \in \mathtt{A}$ with respect to this torsion pair is

$$0 \to \ker(s_{\mathrm{E}}) \to \mathrm{E} \to \mathrm{F} \to 0, \tag{4.1.22}$$

where F is the torsion free part of $\mathrm{H}^0(\mathrm{E})$ in its decomposition $\mathrm{T} \hookrightarrow \mathrm{H}^0(\mathrm{E}) \twoheadrightarrow \mathrm{F}$ with respect to the torsion pair $\langle \mathtt{T}, \mathtt{F} \rangle = \mathrm{Coh}_{\leq 1}(\mathcal{X})$, and $s_{\mathrm{E}}\colon \mathrm{E} \twoheadrightarrow \mathrm{H}^0(\mathrm{E}) \twoheadrightarrow \mathrm{F}$. But then

$$0 \to \mathrm{H}^{-1}(\mathrm{E})[1] \to \ker(s_{\mathrm{E}}) \to \mathrm{T} \to 0 \tag{4.1.23}$$

is an exact sequence in $\mathtt{A}$ whence $\ker(s_{\mathrm{E}}) \in \mathtt{T}'$. We conclude that $\mathtt{F}' = \mathtt{F}$.

Clearly, $\mathtt{T}' \cap \mathtt{G} = \mathtt{V}(\mathtt{T}, \mathtt{F})$. As for the three-term filtration, take an object $\mathrm{E} \in \mathtt{A}$. The first torsion pair induces a unique exact sequence $\mathrm{T}_{\mathrm{E}} \hookrightarrow \mathrm{E} \twoheadrightarrow \mathrm{G}_{\mathrm{E}}$ in $\mathtt{A}$ with $\mathrm{T}_{\mathrm{E}} \in \mathtt{T}$ and $\mathrm{G}_{\mathrm{E}} \in \mathtt{G}$. The second torsion pair induces a unique exact sequence $\mathrm{V}_{\mathrm{E}} \hookrightarrow \mathrm{G}_{\mathrm{E}} \twoheadrightarrow \mathrm{F}_{\mathrm{E}}$ in $\mathtt{A}$ with $\mathrm{V}_{\mathrm{E}} \in \mathtt{T}'$ and $\mathrm{F}_{\mathrm{E}} \in \mathtt{F}$. But $\mathtt{G}$ is closed under subobjects since it is the torsion free part of a torsion pair, so $\mathrm{V}_{\mathrm{E}} \in \mathtt{G}$ as well. It follows that $\mathrm{V}_{\mathrm{E}} \in \mathtt{V}(\mathtt{T}, \mathtt{F})$.

Set $\mathrm{K}_{\mathrm{E}} = \ker(\mathrm{E} \twoheadrightarrow \mathrm{G}_{\mathrm{E}} \twoheadrightarrow \mathrm{F}_{\mathrm{E}})$ in $\mathtt{A}$. The natural three-term filtration of E in $\mathtt{A}$ is $0 \subset \mathrm{T}_{\mathrm{E}} \subset \mathrm{K}_{\mathrm{E}} \subset \mathrm{E}$ with filtration quotients $\mathrm{T}_{\mathrm{E}} \in \mathtt{T}$, $\mathrm{K}_{\mathrm{E}}/\mathrm{T}_{\mathrm{E}} = \mathrm{V}_{\mathrm{E}} \in \mathtt{V}(\mathtt{T}, \mathtt{F})$, and $\mathrm{E}/\mathrm{K}_{\mathrm{E}} = \mathrm{F}_{\mathrm{E}} \in \mathtt{F}$. This proves that $(\mathtt{T}, \mathtt{V}(\mathtt{T}, \mathtt{F}), \mathtt{F})$ defines a torsion triple on $\mathtt{A}$. ☐

Remark 4.1.25. Note that the subcategory of pairs $\mathtt{Pair}(\mathtt{T}, \mathtt{F}) \subset \mathtt{A}$ consists precisely of those objects $\mathrm{E} \in \mathtt{V}(\mathtt{T}, \mathtt{F})$ that satisfy $\mathrm{rk}(\mathrm{E}) = -1$.

We record the following easy lemma for later use.

Lemma 4.1.26. Let $(\mathtt{T}, \mathtt{F})$ be a torsion pair on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$, let E be a $(\mathtt{T}, \mathtt{F})$-pair of the form $\mathrm{E} = (\mathcal{O}_{\mathcal{X}} \to \mathrm{G})$, and let $\mathrm{C} \in \mathrm{Coh}_{\leq 1}(\mathcal{X})$ be an object. If C is a subobject of E in $\mathtt{A}$, then the inclusion factors through an inclusion $\mathrm{C} \hookrightarrow \mathrm{G}$.

Proof. The $(\mathtt{T}, \mathtt{F})$-pair E fits into the short exact sequence $0 \to \mathrm{G} \to \mathrm{E} \to \mathcal{O}_{\mathcal{X}}[1] \to 0$ in $\mathtt{A}$. By Serre duality $\mathrm{Hom}(\mathrm{C}, \mathcal{O}_{\mathcal{X}}[1]) = \mathrm{H}^2(\mathcal{X}, \mathrm{C})^{\vee} = 0$, so the claim follows. ☐

4.2 Moduli and boundedness of pairs

Let $(\mathtt{T},\mathtt{F})$ denote a torsion pair on $\mathrm{Coh}_{\le 1}(\mathcal{X})$, and let $\mathtt{A}$ denote Toda's category from Definition 4.1.4. Recall that $\mathtt{Pair}(\mathtt{T},\mathtt{F}) \subset \mathtt{A}$ denotes the full subcategory of $(\mathtt{T},\mathtt{F})$-pairs.

We define the moduli stack of pairs as a substack of M. Lieblich's moduli stack $\mathfrak{Mum}_{\mathcal{X}}$ as defined in section 2.2.2. However, this stack is only shown to be algebraic and locally of finite type for smooth proper varieties. We first show that the McKay equivalence $\Phi\colon \mathrm{D}(\mathrm{Y}) \to \mathrm{D}(\mathcal{X})$ induces an isomorphism of stacks $\Phi\colon \mathfrak{Mum}_{\mathrm{Y}} \to \mathfrak{Mum}_{\mathcal{X}}$, proving that the latter is also algebraic and locally of finite type. This result is not surprising as these stacks parametrise objects in hearts on the *same* triangulated category.

Then we define the moduli stack of pairs as a substack of $\mathfrak{Mum}_{\mathcal{X}}$. Under some reasonable conditions on the torsion pair, the moduli stack of pairs defines an open substack. As a direct consequence of this construction, it is then an algebraic stack locally of finite type. Moreover, for torsion pairs induced by a stability condition on $\mathrm{Coh}_{\le 1}(\mathcal{X})$, this stack is a $\mathbf{C}^{\times}$-gerbe over its coarse moduli space. In particular, we obtain an element in the motivic Hall algebra $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$.

Finally, we define the virtual count of $(\mathtt{T},\mathtt{F})$-pairs of a fixed numerical class by applying the integration morphism of 2.3.26 to this element.

4.2.1 Moduli of complexes

Recall the McKay equivalence $\Phi\colon \mathrm{D}(\mathrm{Y}) \to \mathrm{D}(\mathcal{X})$ of Theorem 2.4.11. It is defined by a Fourier–Mukai transform with kernel the Y-flat sheaf $\mathcal{O}_{\mathcal{Z}}$ on $\mathcal{X} \times \mathrm{Y}$, where $\mathcal{Z} \subset \mathcal{X} \times \mathrm{Y}$ is the universal closed subscheme parametrising non-stacky points on $\mathcal{X}$ as quotients of $\mathcal{O}_{\mathcal{X}}$. So

$$\Phi(\mathrm{E}) = \mathbf{R}\pi_{\mathcal{X},*}(\mathcal{O}_{\mathcal{Z}} \overset{\mathbf{L}}{\otimes} \pi_{\mathrm{Y}}^{*}(\mathrm{E})) \tag{4.2.1}$$

for $\mathrm{E} \in \mathrm{D}(\mathrm{Y})$, where $\pi_{\mathcal{X}}\colon \mathcal{X} \times \mathrm{Y} \to \mathcal{X}$ and $\pi_{\mathrm{Y}}\colon \mathcal{X} \times \mathrm{Y} \to \mathrm{Y}$ denote the natural projections.

Fourier–Mukai transforms behave well in families. Indeed, let S be a scheme, let

$$\mathrm{S} \times \mathrm{Y} \xleftarrow{\pi_{\mathrm{S}\times\mathrm{Y}}} \mathrm{S} \times \mathcal{X} \times \mathrm{Y} \xrightarrow{\pi_{\mathrm{S}\times\mathcal{X}}} \mathrm{S} \times \mathcal{X} \tag{4.2.2}$$

denote the natural projections, and write $p_{\mathrm{S}}\colon \mathrm{S} \times \mathcal{X} \times \mathrm{Y} \to \mathcal{X} \times \mathrm{Y}$. Let $\mathrm{F}\colon \mathrm{D}(\mathrm{Y}) \to \mathrm{D}(\mathcal{X})$ be a Fourier–Mukai functor, i.e., a functor as in equation (4.2.1) with $\mathcal{O}_{\mathcal{Z}}$ replaced by any *kernel* $\mathcal{P} \in \mathrm{D}(\mathcal{X} \times \mathrm{Y})$. Such F induces a relative Fourier–Mukai functor

$$\mathrm{F}_{\mathrm{S}}(\mathrm{E}) \coloneqq \mathbf{R}\pi_{\mathcal{X}\times\mathrm{S},*}(p_{\mathrm{S}}^{*}(\mathcal{P}) \overset{\mathbf{L}}{\otimes} \pi_{\mathrm{Y}\times\mathrm{S}}^{*}(\mathrm{E})) \tag{4.2.3}$$

sending $\mathrm{E} \in \mathrm{D}(\mathrm{S} \times \mathrm{Y})$ to an object of $\mathrm{D}(\mathrm{S} \times \mathcal{X})$. We have the following

Lemma 4.2.1. Let $\mathrm{E} \in \mathrm{D}(\mathrm{S} \times \mathrm{Y})$ be S-perfect. Then $\mathrm{F}_{\mathrm{S}}(\mathrm{E}) \in \mathrm{D}(\mathrm{S} \times \mathcal{X})$ is S-perfect and it satisfies $\mathrm{F}_{\mathrm{S}}(\mathrm{E})_s = \mathrm{F}(\mathrm{E}_s)$ for all closed points $s \in \mathrm{S}$.

Recall that $\mathrm{E}_s = \mathbf{L}i_s^*(\mathrm{E})$ denotes the derived restriction of E. When both $\mathcal{X}$ and Y are smooth projective varieties, this result is well-known; see [Bri99, Lem. 4.1].

Proof. Recall that the notions of S-perfect and perfect coincide since $\mathrm{S} \times \mathcal{X}$ and $\mathrm{S} \times \mathrm{Y}$ are smooth over S; see Remark 2.2.10. By assumption, $\mathrm{E}_s \in \mathrm{D}(\mathrm{Y})$ so $\mathrm{F}(\mathrm{E}_s) \in \mathrm{D}(\mathcal{X})$ for all $s \in \mathrm{S}$. Thus it suffices to prove that $\mathrm{F}_{\mathrm{S}}(\mathrm{E})_s = \mathrm{F}(\mathrm{E}_s)$ for all closed points $s \in \mathrm{S}$.

We prove the following more general base change result with some care. Let $g\colon \mathrm{T} \to \mathrm{S}$ be a morphism of arbitrary schemes, write $g_{\mathrm{Y}} = g \times \mathbf{1}_{\mathrm{Y}}\colon \mathrm{T} \times \mathrm{Y} \to \mathrm{S} \times \mathrm{Y}$, and similarly write $g_{\mathcal{X}} = g \times \mathbf{1}_{\mathcal{X}}$. Consider the Cartesian diagram

$$\begin{tikzcd}
\mathrm{T} \times \mathrm{Y} & \mathrm{T} \times \mathcal{X} \times \mathrm{Y} \arrow[l, "\pi_{\mathrm{T}\times\mathrm{Y}}"'] \arrow[r, "g_{\mathcal{X}\times\mathrm{Y}}"] \arrow[d, "\pi_{\mathrm{T}\times\mathcal{X}}"'] \arrow[dr, phantom, "\square"] & \mathrm{S} \times \mathcal{X} \times \mathrm{Y} \arrow[r, "\pi_{\mathrm{S}\times\mathrm{Y}}"] \arrow[d, "\pi_{\mathrm{S}\times\mathcal{X}}"] & \mathrm{S} \times \mathrm{Y} \\
& \mathrm{T} \times \mathcal{X} \arrow[r, "g_{\mathcal{X}}"] & \mathrm{S} \times \mathcal{X} &
\end{tikzcd} \tag{4.2.4}$$

Consider the canonical base change morphism of functors

$$b_{\mathcal{X}}\colon \mathbf{L}g_{\mathcal{X}}^* \circ \mathbf{R}\pi_{\mathrm{S}\times\mathcal{X},*} \to \mathbf{R}\pi_{\mathrm{T}\times\mathcal{X},*} \circ \mathbf{L}g_{\mathcal{X}\times\mathrm{Y}}^*. \tag{4.2.5}$$

Both sides map $\mathrm{Perf}(\mathrm{S} \times \mathcal{X} \times \mathrm{Y})$ to $\mathrm{Perf}(\mathrm{T} \times \mathcal{X})$. Indeed, the derived pullback of a perfect complex is again perfect, and the same holds for the derived pushforward over the smooth projective morphisms $\pi_{\mathrm{T}\times\mathcal{X}}$ and $\pi_{\mathrm{S}\times\mathcal{X}}$. For $\mathrm{G} \in \mathrm{Perf}(\mathrm{S} \times \mathcal{X} \times \mathrm{Y})$, note that $b_{\mathcal{X}}(\mathrm{G})$ is an isomorphism if and only if it induces isomorphisms on cohomology sheaves. This may be checked on an étale cover, so let $a\colon \mathrm{A} \twoheadrightarrow \mathcal{X}$ be an atlas of $\mathcal{X}$, i.e., A is a smooth threefold and a is an étale surjection. Consider the canonical morphism of functors

$$a_{\mathrm{T}}^* \circ b_{\mathcal{X}}\colon (a_{\mathrm{T}}^* \circ \mathbf{L}g_{\mathcal{X}}^* \circ \mathbf{R}\pi_{\mathrm{S}\times\mathcal{X},*}) \to (a_{\mathrm{T}}^* \circ \mathbf{R}\pi_{\mathrm{T}\times\mathcal{X},*} \circ \mathbf{L}g_{\mathcal{X}\times\mathrm{Y}}^*) \tag{4.2.6}$$

where $a_{\mathrm{T}} = \mathbf{1}_{\mathrm{T}} \times a$. Consider the following diagram in which all squares are Cartesian.

$$\begin{tikzcd}
\mathrm{T} \times \mathrm{A} \times \mathrm{Y} \arrow[rrr, "g_{\mathrm{A}\times\mathrm{Y}}"] \arrow[dr, two heads, "a_{\mathrm{T}\times\mathrm{Y}}"'] \arrow[ddd, "\pi_{\mathrm{T}\times\mathrm{A}}"'] & & & \mathrm{S} \times \mathrm{A} \times \mathrm{Y} \arrow[dl, two heads, "a_{\mathrm{S}\times\mathrm{Y}}"] \arrow[ddd, "\pi_{\mathrm{S}\times\mathrm{A}}"] \\
& \mathrm{T} \times \mathcal{X} \times \mathrm{Y} \arrow[r, "g_{\mathcal{X}\times\mathrm{Y}}"] \arrow[d, "\pi_{\mathrm{T}\times\mathcal{X}}"'] \arrow[dr, phantom, "\square"] & \mathrm{S} \times \mathcal{X} \times \mathrm{Y} \arrow[d, "\pi_{\mathrm{S}\times\mathcal{X}}"] & \\
& \mathrm{T} \times \mathcal{X} \arrow[r, "g_{\mathcal{X}}"] & \mathrm{S} \times \mathcal{X} & \\
\mathrm{T} \times \mathrm{A} \arrow[ur, two heads, "a_{\mathrm{T}}"] \arrow[rrr, "g_{\mathrm{A}}"] & & & \mathrm{S} \times \mathrm{A} \arrow[ul, two heads, "a_{\mathrm{S}}"']
\end{tikzcd} \tag{4.2.7}$$

Since a_{T} and a_{S} are étale, hence flat, and since $\pi_{\mathrm{T}\times\mathrm{A}}$ and $\pi_{\mathrm{S}\times\mathrm{A}}$ are smooth projective morphisms, the left and right squares satisfy flat base change by [Nir08, Thm. 1.9]. It follows that the morphism $a_{\mathrm{T}}^{*} \circ b_{\mathcal{X}}$ in equation (4.2.6) is identified with the morphism

$$b_{\mathrm{A}} \circ a_{\mathrm{S}\times\mathrm{Y}}^{*} \colon (\mathbf{L}g_{\mathcal{X}}^{*} \circ \mathbf{R}\pi_{\mathrm{S}\times\mathcal{X},*}) \circ a_{\mathrm{S}\times\mathrm{Y}}^{*} \to (\mathbf{R}\pi_{\mathrm{T}\times\mathcal{X},*} \circ \mathbf{L}g_{\mathcal{X}\times\mathrm{Y}}^{*}) \circ a_{\mathrm{S}\times\mathrm{Y}}^{*}. \tag{4.2.8}$$

The canonical base change morphism b_{A} around the outer square is an isomorphism because $\pi_{\mathrm{S}\times\mathrm{A}}$ is smooth, hence flat, so the base change theorem for schemes applies. Take $\mathrm{G} \in \mathrm{Perf}(\mathrm{S} \times \mathcal{X} \times \mathrm{Y})$. It follows that the morphisms on cohomology sheaves

$$a_{\mathrm{T}}^{*}\mathrm{H}_{\mathrm{T}\times\mathcal{X}}^{i}\big(b_{\mathcal{X}}(\mathrm{G})\big) = \mathrm{H}_{\mathrm{T}\times\mathrm{A}}^{i}\big((a_{\mathrm{T}}^{*} \circ b_{\mathcal{X}})(\mathrm{G})\big) = \mathrm{H}_{\mathrm{T}\times\mathrm{A}}^{i}\big(b_{\mathrm{A}}(a_{\mathrm{S}\times\mathrm{Y}}^{*}(\mathrm{G}))\big) \tag{4.2.9}$$

are isomorphisms for all i. We conclude that $b_{\mathcal{X}}(\mathrm{G})$ is an isomorphism as claimed.

Let $\mathrm{E} \in \mathrm{D}(\mathrm{S} \times \mathrm{Y})$ be an S-perfect complex. It follows that

$$(\mathbf{L}g_{\mathcal{X}}^{*} \circ \mathrm{F}_{\mathrm{S}})(\mathrm{E}) = (\mathrm{F}_{\mathrm{T}} \circ \mathbf{L}g_{\mathrm{Y}}^{*})(\mathrm{E}), \tag{4.2.10}$$

by stacky base change around the diagram 4.2.4; this is the analogue of [Bri99, Lem. 4.1]. Applying this result to the closed immersions $g = i_s \colon \{s\} \hookrightarrow \mathrm{S}$ shows that $\mathrm{F}_{\mathrm{S}}(\mathrm{E})_s = \mathrm{F}(\mathrm{E}_s)$ as claimed. This completes the proof. □

We are now in a position to prove the following

Proposition 4.2.2. The McKay equivalence $\Phi \colon \mathrm{D}(\mathrm{Y}) \to \mathrm{D}(\mathcal{X})$ induces an isomorphism

$$\Phi \colon \mathfrak{Mum}_{\mathrm{Y}} \to \mathfrak{Mum}_{\mathcal{X}} \tag{4.2.11}$$

of stacks, where an S-valued point $\mathrm{E} \in \mathfrak{Mum}_{\mathrm{Y}}(\mathrm{S})$ is sent to $\Phi_{\mathrm{S}}(\mathrm{E}) \in \mathfrak{Mum}_{\mathcal{X}}(\mathrm{S})$.

Proof. Let S be a scheme, and let $\mathrm{E} \in \mathfrak{Mum}_{\mathrm{Y}}(\mathrm{S})$ be an S-valued point. By the previous lemma, we have $\Phi_{\mathrm{S}}(\mathrm{E})_s = \Phi(\mathrm{E}_s) \in \mathrm{D}(\mathcal{X})$ for all geometric points $s \in \mathrm{S}$, so $\Phi_{\mathrm{S}}(\mathrm{E})$ is S-perfect. By assumption $\mathrm{Ext}_{\mathrm{Y}_s}^{i}(\mathrm{E}_s, \mathrm{E}_s) = 0$ for all geometric points $s \in \mathrm{S}$ and $i < 0$. We deduce that

$$\mathrm{Ext}_{\mathcal{X}_s}^{i}(\Phi_{\mathrm{S}}(\mathrm{E})_s, \Phi_{\mathrm{S}}(\mathrm{E})_s) = \mathrm{Hom}_{\mathcal{X}_s}^{i}(\Phi(\mathrm{E}_s), \Phi(\mathrm{E}_s)) = \mathrm{Hom}_{\mathrm{Y}_s}^{i}(\mathrm{E}_s, \mathrm{E}_s) = 0 \tag{4.2.12}$$

because Φ is an equivalence. It follows that $\Phi_{\mathrm{S}}(\mathrm{E}) \in \mathfrak{Mum}_{\mathcal{X}}(\mathrm{S})$ as claimed.

We claim that $\Phi \colon \mathfrak{Mum}_{\mathrm{Y}} \to \mathfrak{Mum}_{\mathcal{X}}$ is an isomorphism. Equivalently, we claim that $\Phi_{\mathrm{S}} \colon \mathfrak{Mum}_{\mathrm{Y}}(\mathrm{S}) \to \mathfrak{Mum}_{\mathcal{X}}(\mathrm{S})$ is an equivalence of groupoids for any scheme S. This follows because the left (and right) adjoint of Φ_{S}, i.e., the S-relative Fourier–Mukai transform with kernel $p_{\mathrm{S}}^{*}\,\mathbf{R}\underline{\mathrm{Hom}}(\mathcal{O}_{\mathcal{Z}}, \mathcal{O}_{\mathcal{X}\times\mathrm{Y}})[3]$, is the inverse. □

Corollary 4.2.3. The stack $\mathfrak{Mum}_{\mathcal{X}}$ is an algebraic stack locally of finite type. It decomposes as a disjoint union into open and closed substacks

$$\mathfrak{Mum}_{\mathcal{X}} = \bigsqcup_{\alpha \in \mathrm{N}(\mathcal{X})} \mathfrak{Mum}_{\mathcal{X},\alpha} \tag{4.2.13}$$

where $\mathfrak{Mum}_{\mathcal{X},\alpha}$ parametrises objects of class α in $\mathrm{N}(\mathcal{X})$.

Proof. This is immediate as it holds for $\mathfrak{Mum}_{\mathrm{Y}}$ by Theorem 2.2.12. □

Remark 4.2.4. We may transfer all results of section 2.2.2 to the stack $\mathfrak{Mum}_{\mathcal{X}}$.

Finally, we collect a similar statement for the derived dual $\mathbf{D} = \mathbf{R}\underline{\mathrm{Hom}}(-, \mathcal{O}_{\mathcal{X}})$. For a scheme S we denote $\mathbf{D}_{\mathrm{S}}(\mathrm{E}) = \mathbf{R}\underline{\mathrm{Hom}}(\mathrm{E}, \mathcal{O}_{\mathrm{S}\times\mathcal{X}})$ for the derived dual on $\mathrm{D}(\mathrm{S} \times \mathcal{X})$.

Proposition 4.2.5. The derived dual induces an automorphism of the stack $\mathfrak{Mum}_{\mathcal{X}}$, where an S-valued point $\mathrm{E} \in \mathfrak{Mum}_{\mathcal{X}}(\mathrm{S})$ is sent to $\mathbf{D}_{\mathrm{S}}(\mathrm{E}) \in \mathfrak{Mum}_{\mathcal{X}}(\mathrm{S})$.

Note that $\mathbf{D}$ defines an anti-equivalence on $\mathrm{D}(\mathcal{X})$. So it should be a Fourier–Mukai functor by an analogue of D. Orlov's theorem for orbifolds with projective coarse space. The result then follows from Lemma 4.2.1. For lack of a reference, we present a proof.

Proof. We follow the strategy of the proof of Lemma 4.2.1. Let S be a scheme and let $\mathrm{E} \in \mathfrak{Mum}_{\mathcal{X}}(\mathrm{S})$ be an S-valued point. For an S-perfect complex, taking derived fibres commutes with taking the derived dual. In particular, $\mathbf{D}_{\mathrm{S}}(\mathrm{E})_s = \mathbf{D}(\mathrm{E}_s) \in \mathrm{D}(\mathcal{X})$ shows that $\mathbf{D}_{\mathrm{S}}(\mathrm{E})$ is again S-perfect. Moreover, it follows that

$$\mathrm{Ext}^i_{\mathcal{X}}(\mathbf{D}_{\mathrm{S}}(\mathrm{E})_s, \mathbf{D}_{\mathrm{S}}(\mathrm{E})_s) = \mathrm{Ext}^i_{\mathcal{X}}(\mathbf{D}(\mathrm{E}_s), \mathbf{D}(\mathrm{E}_s)) = \mathrm{Ext}^i_{\mathcal{X}}(\mathrm{E}_s, \mathrm{E}_s) = 0 \tag{4.2.14}$$

for every geometric point $s \in \mathrm{S}$ and all $i < 0$. We conclude that $\mathbf{D}_{\mathrm{S}}(\mathrm{E}) \in \mathfrak{Mum}_{\mathcal{X}}(\mathrm{S})$. Furthermore, the natural morphism $\mathrm{E} \to \mathbf{D}_{\mathrm{S}}(\mathbf{D}_{\mathrm{S}}(\mathrm{E}))$ is an isomorphism. This shows that $\mathbf{D}_{\mathrm{S}}$ is its own inverse, and thus completes the proof. □

4.2.2 Moduli of pairs

Recall our convention of Remark 2.2.16 to denote by $\underline{\mathtt{C}}$ the stack parametrising objects in a subcategory $\mathtt{C} \subset \mathfrak{Mum}_{\mathcal{X}}$. Given a torsion pair $(\mathtt{T}, \mathtt{F})$ on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$, we now explain the construction of the stack $\underline{\mathtt{Pair}}(\mathtt{T}, \mathtt{F})$ as a substack of $\mathfrak{Mum}_{\mathcal{X}}$.

First, recall that $\mathtt{A}$ is constructed as a full subcategory of the tilt

$$\mathrm{Coh}^\flat(\mathcal{X}) = \langle \mathrm{Coh}_{\geq 2}(\mathcal{X})[1], \mathrm{Coh}_{\leq 1}(\mathcal{X}) \rangle \subset \mathrm{D}^{[-1,0]}(\mathcal{X}). \tag{4.2.15}$$

As such, the objects of $\mathtt{A}$ lie in the heart of a bounded t-structure on $\mathrm{D}(\mathcal{X})$ and, hence, are $\mathbf{C}$-valued points of $\mathfrak{Mum}_{\mathcal{X}}$. There is the following result.

Lemma 4.2.6. The moduli stack $\underline{\mathrm{Coh}}^\flat_{\mathcal{X}} \subset \mathfrak{Mum}_{\mathcal{X}}$ parametrising objects in $\mathrm{Coh}^\flat(\mathcal{X})$ defines an open substack. In particular, it is an algebraic stack locally of finite type.

Proof. The pair $(\mathrm{Coh}_{\leq 1}(\mathcal{X}), \mathrm{Coh}_{\geq 2}(\mathcal{X}))$ defines a stack of open torsion theories in the sense of [AB13, App. A]. The result follows by the argument of [AB13, Thm. A.3]. □

Second, recall the subcategory $\mathtt{S} \subset \mathtt{A}$ parametrising objects of 'small' rank, i.e., objects $\mathrm{E} \in \mathtt{A}$ such that $\mathrm{rk}(\mathrm{E}) = -1, 0$ as characterised in Proposition 4.1.8. We realise its moduli stack $\underline{\mathtt{S}}$ as an open substack of $\underline{\mathrm{Coh}}^\flat_{\mathcal{X}}$.

Lemma 4.2.7. The substack $\underline{\mathtt{S}} \subset \underline{\mathrm{Coh}}^\flat(\mathcal{X})$ parametrising objects in $\mathtt{S}$ defines an open substack. In particular, it is an algebraic stack locally of finite type.

Proof. The stack $\underline{\mathtt{S}}$ splits as a disjoint union according to rank. The rank zero component is $\underline{\mathrm{Coh}}_{\leq 1,\mathcal{X}}$, which is open in $\underline{\mathrm{Coh}}^\flat_{\mathcal{X}}$ as it is the intersection

$$\underline{\mathrm{Coh}}_{\leq 1,\mathcal{X}} = \underline{\mathrm{Coh}}_{\mathcal{X}} \cap \underline{\mathrm{Coh}}_{\flat,\mathcal{X}} \tag{4.2.16}$$

of two open substack inside $\mathfrak{Mum}_{\mathcal{X}}$. For the rank minus one component, we appeal to the characterisation of these objects given in Proposition 4.1.8. Let S be a base scheme, which we may assume to be of finite type over $\mathbf{C}$, and let $\mathrm{E} \in \underline{\mathrm{Coh}}^\flat_{\mathcal{X}}(\mathrm{S})$. We prove that the locus $\mathrm{U} \subset \mathrm{S}$ for which $\mathrm{E}_s \in \mathtt{S}$ is open in S. Concretely, $s \in \mathrm{U}$ if and only if $\mathrm{H}^{-1}(\mathrm{E}_s)$ is torsion free with trivial determinant; we refer to this condition as TFTD. To show that $\mathrm{U} \subset \mathrm{S}$ is open, we again follow [AB13, Thm. A.3].

We claim that having the property TFTD is open in flat families. Using the argument of [HL10, Prop 2.3.1] combined with the appropriate stacky version of the Grothendieck Lemma [Nir08, Lem. 4.13], it follows that being torsion free is open in flat families.

As for the triviality of the determinant, let T be a base scheme. Since $\mathcal{X}$ is smooth, any $\mathrm{E} \in \mathrm{D}(\mathcal{X}) = \mathrm{Perf}(\mathcal{X})$ is quasi-isomorphic to a bounded complex L of locally free sheaves. Then

$$\det(\mathrm{E}) = \bigotimes_i \det(\mathrm{L}_i)^{(-1)^i}. \tag{4.2.17}$$

Similarly, any T-perfect complex $\mathrm{E} \in \mathrm{D}(\mathrm{T} \times \mathcal{X})$ is in fact perfect, is thus quasi-isomorphic to a bounded complex of locally free sheaves on $\mathrm{T} \times \mathcal{X}$, and its determinant is defined in the same way. Forming the determinant commutes with taking derived fibres, so

$$(\det \mathrm{E})_t \cong \det(\mathrm{E}_t) \tag{4.2.18}$$

for all closed points $t \in \mathrm{T}$. In particular, given a T-flat family of coherent sheaves on $\mathcal{X}$, we conclude that having a trivial determinant is an open property since $\mathrm{H}^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = 0$, i.e., since the structure sheaf $\mathcal{O}_{\mathcal{X}}$ is rigid. This proves the claim.

We apply this result twice in order to show that $U \subset S$ is open. Openness of U is equivalent to U being constructible and stable under generization. For constructibility one may pass to an S-flattening stratification of $H^{-1}(E)$. Let H denote the restriction of $H^{-1}(E)$ to a given stratum $T \subset S$. By assumption, H is T-flat so the locus where H is TFTD is open. This shows that $U \cap T$ is open, and we conclude that U is constructible.

To prove that U is stable under generization, we may assume that S is the spectrum of a discrete valuation ring, with closed point s and generic point η. Assume that $H^{-1}(E)_s$ is TFTD. Let t be a uniformizer. We have an exact sequence in $\mathrm{Coh}(S \times \mathcal{X})$:

$$0 \to H^{-1}(E) \xrightarrow{\cdot t} H^{-1}(E) \to H^{-1}(E_s) \to H^0(E) \xrightarrow{\cdot t} H^0(E) \to H^0(E_s) \to 0.$$

Since multiplication by t is injective, $H^{-1}(E)$ is flat over S. Again, since being TFTD is open in flat families, it follows that $H^{-1}(E)_\eta$ is TFTD and we may conclude. □

Third, we give conditions when $\underline{\mathtt{Pair}}(\mathtt{T},\mathtt{F})$ is an open substack of $\underline{\mathtt{S}}$. In order to do so, we need two lemmas and a definition.

Lemma 4.2.8. Let $\mathtt{T} \subset \mathrm{Coh}_{\leq 1}(\mathcal{X})$ be a subcategory closed under quotients. Suppose that the corresponding moduli stack $\underline{\mathtt{T}} \subset \underline{\mathrm{Coh}}_{\leq 1,\mathcal{X}}$ is open. Then the locus in $\underline{\mathtt{S}}$ of those E such that $H^0(E) \in \mathtt{T}$ is open.

Note that the case $\mathtt{T} = 0$ is also covered by this lemma.

Proof. This is the first half of the proof in Lieblich's appendix [Lie06]. □

Definition 4.2.9. A torsion pair $(\mathtt{T},\mathtt{F})$ on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ is called *open* if the substacks $\underline{\mathtt{T}}, \underline{\mathtt{F}} \subset \underline{\mathrm{Coh}}_{\leq 1}(\mathcal{X})$ are open.

Let $(\mathtt{T},\mathtt{F})$ be an open torsion pair on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$. An object $E \in \mathtt{S}$ is a $(\mathtt{T},\mathtt{F})$-pair if three conditions are satisfied:

1. $\mathrm{rk}(E) = -1$, which is open in flat families by the proof of Proposition 4.1.8;
2. $H^0(E) \in \mathtt{T}$, which is open in flat families by virtue of $(\mathtt{T},\mathtt{F})$ being open;
3. $\mathrm{Hom}(T,E) = 0$ for all $T \in \mathtt{T}$.

To show that the latter condition is also open in flat families, we reformulate it in terms of a condition on the derived dual of E that is clearly open; this is sufficient by Proposition 4.2.5.Henceforth, by *derived dual* we mean the anti-equivalence of $D(\mathcal{X})$ given by

$$\mathbf{D}(-) = \mathbf{R}\underline{\mathrm{Hom}}(-,\mathcal{O}_{\mathcal{X}})[2]. \tag{4.2.19}$$

Note that the shift by two is added as opposed to the notion of $\mathbf{D}$ in Proposition 4.2.5.

Moreover, recall that $\mathbf{D}(\mathrm{Coh}_1(\mathcal{X})) = \mathrm{Coh}_1(\mathcal{X})$, $\mathbf{D}(\mathrm{Coh}_0(\mathcal{X})) = \mathrm{Coh}_0(\mathcal{X})[-1]$ because $\mathcal{X}$ is three-dimensional. In particular, it follows that the abelian category $\mathbf{D}(\mathrm{Coh}_{\le 1}(\mathcal{X}))$ has a torsion pair given by $\mathbf{D}(\mathrm{Coh}_{\le 1}(\mathcal{X})) = (\mathrm{Coh}_1(\mathcal{X}), \mathrm{Coh}_0(\mathcal{X})[-1])$.

Lemma 4.2.10. Let $\mathrm{E} \in \mathtt{A}$ be an object of $\mathrm{rk}(\mathrm{E}) = -1$. Its derived dual $\mathbf{D}(\mathrm{E})$ is a complex concentrated in degrees $-1, 0, 1$ and $\mathrm{H}^1(\mathbf{D}(\mathrm{E})) \in \mathrm{Coh}_0(\mathcal{X})$.

Proof. Recall that $\mathrm{H}^{-1}(\mathrm{E})$ is torsion free and that $\mathrm{H}^0(\mathrm{E}) \in \mathrm{Coh}_{\le 1}(\mathcal{X})$ by Proposition 4.1.8. It follows that $\mathbf{D}(\mathrm{H}^{-1}(\mathrm{E})[1]) \in \mathrm{D}^{[-1,1]}(\mathcal{X})$ and $\mathbf{D}(\mathrm{H}^0(\mathrm{E})) \in \mathrm{D}^{[0,1]}(\mathcal{X})$. Moreover, both $\mathrm{H}^1(\mathbf{D}(\mathrm{H}^{-1}(\mathrm{E})[1]))$ and $\mathrm{H}^1(\mathbf{D}(\mathrm{H}^0(\mathrm{E})))$ are zero-dimensional. The claims now follow from the exact triangle $\mathrm{H}^{-1}(\mathrm{E})[1] \to \mathrm{E} \to \mathrm{H}^0(\mathrm{E})$ in $\mathrm{D}(\mathcal{X})$. ☐

We wish to compare notions of pair with respect to different torsion pairs. In order to do so, the following lemma is stated in the appropriate generality.

Lemma 4.2.11. Let $(\mathtt{T}, \mathtt{F})$, $(\tilde{\mathtt{T}}, \tilde{\mathtt{F}})$ be two torsion pairs on $\mathrm{Coh}_{\le 1}(\mathcal{X})$. Assume that $\mathrm{Coh}_0(\mathcal{X}) \subset \mathtt{T}$, so $\mathtt{F} \subset \mathrm{Coh}_1(\mathcal{X})$. An object $\mathrm{E} \in \mathtt{A}$ of $\mathrm{rk}(\mathrm{E}) = -1$ is a $(\mathtt{T}, \tilde{\mathtt{F}})$-pair if and only if the following two conditions holds:

1. $\mathrm{H}^0(\mathrm{E}) \in \tilde{\mathtt{T}}$,
2. $\mathrm{H}^1(\mathbf{D}(\mathrm{E})) = 0$ and $\mathrm{H}^0(\mathbf{D}(\mathrm{E})) \in \langle \mathrm{Coh}_0(\mathcal{X}), \mathbf{D}(\mathtt{F})\rangle$.

Proof. Let $\mathrm{E} \in \mathtt{A}$ be an object of $\mathrm{rk}(\mathrm{E}) = -1$. We see that $\mathrm{Hom}(\mathrm{E}, \tilde{\mathtt{F}}) = 0$ is equivalent to $\mathrm{H}^0(\mathrm{E}) \in \tilde{\mathtt{T}}$ by applying the functor $\mathrm{Hom}(-, \tilde{\mathtt{F}})$ to the short exact sequence

$$0 \to \mathrm{H}^{-1}(\mathrm{E})[1] \to \mathrm{E} \to \mathrm{H}^0(\mathrm{E}) \to 0 \tag{4.2.20}$$

in $\mathtt{A}$. Here we use the fact that $\mathrm{Hom}(\mathrm{H}^{-1}(\mathrm{E})[1], \tilde{\mathtt{F}}) = \mathrm{Ext}^{-1}_{\mathcal{X}}(\mathrm{H}^{-1}(\mathrm{E}), \tilde{\mathtt{F}}) = 0$ since there are no negative extensions between objects of $\mathrm{Coh}(\mathcal{X})$.

Denote $\mathtt{T}_1 = \mathtt{T} \cap \mathrm{Coh}_1(\mathcal{X})$. Since $\mathrm{Coh}_0(\mathcal{X}) \subset \mathtt{T}$, the condition $\mathrm{Hom}(\mathtt{T}, \mathrm{E}) = 0$ is equivalent to $\mathrm{Hom}(\mathrm{Coh}_0(\mathcal{X}), \mathrm{E}) = 0$ and $\mathrm{Hom}(\mathtt{T}_1, \mathrm{E}) = 0$. We have a torsion triple

$$\mathrm{Coh}_{\le 1}(\mathcal{X}) = \langle \mathrm{Coh}_0(\mathcal{X}), \mathtt{T}_1, \mathtt{F}\rangle \tag{4.2.21}$$

since $\mathtt{F} \subset \mathrm{Coh}_1(\mathcal{X})$. Note that no object in $\mathtt{A}$ admits a non-zero map to $\mathrm{Coh}_0(\mathcal{X})[-1]$. This exposes the symmetry of pairs under dualising: $\mathrm{Hom}(\mathtt{T}, \mathrm{E}) = 0$ holds if and only if $\mathrm{Hom}(\mathbf{D}(\mathrm{E}), \mathrm{Coh}_0(\mathcal{X})[-1])) = 0$ and $\mathrm{Hom}(\mathbf{D}(\mathrm{E}), \mathbf{D}(\mathtt{T}_1)) = 0$. These two conditions are then equivalent to the two conditions $\mathrm{H}^1(\mathbf{D}(\mathrm{E})) = 0$ and $\mathrm{H}^0(\mathbf{D}(\mathrm{E})) \in \langle \mathrm{Coh}_0(\mathcal{X}), \mathbf{D}(\mathtt{F})\rangle$ by Lemma 4.2.10. ☐

We now prove the openness of $\underline{\mathtt{Pair}}(\mathtt{T}, \mathtt{F}) \subset \mathfrak{Mum}_{\mathcal{X}}$ for an open torsion pair.

Proposition 4.2.12. Let $(\mathtt{T},\mathtt{F})$, $(\tilde{\mathtt{T}},\tilde{\mathtt{F}})$ be two open torsion pairs on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$. Assume that $\mathrm{Coh}_0(\mathcal{X})\subset\mathtt{T}$, so $\mathtt{F}\subset\mathrm{Coh}_1(\mathcal{X})$. The substack $\underline{\mathtt{Pair}}(\mathtt{T},\tilde{\mathtt{F}})\subset\mathfrak{Mum}_{\mathcal{X}}$ parametrising $(\mathtt{T},\tilde{\mathtt{F}})$-pairs is open. In particular, it is an algebraic stack locally of finite type.

Proof. By Proposition 4.2.5, the shifted duality functor $\mathbf{D}$ induces an automorphism of the stack $\mathfrak{Mum}_{\mathcal{X}}$. So if $\mathrm{G}\in\mathfrak{Mum}_{\mathcal{X}}(\mathrm{S})$ is an S-perfect family of complexes over some base scheme S, then so is its dual $\mathbf{D}(\mathrm{G})=\mathbf{R}\underline{\mathrm{Hom}}(\mathrm{G},\mathcal{O}_{\mathrm{S}\times\mathcal{X}})[2]\in\mathfrak{Mum}_{\mathcal{X}}(\mathrm{S})$.

An object E of $\mathrm{rk}(\mathrm{E})=-1$ is a $(\mathtt{T},\tilde{\mathtt{F}})$-pair if and only if three properties hold: (i) $\mathrm{H}^0(\mathrm{E})\in\tilde{\mathtt{T}}$, (ii) $\mathrm{H}^1(\mathbf{D}(\mathrm{E}))=0$, and (iii) $\mathrm{H}^0(\mathbf{D}(\mathrm{E}))\in\mathtt{G}:=\langle\mathrm{Coh}_0(\mathcal{X}),\mathbf{D}(\mathtt{F})\rangle$. By Lemma 4.2.8, the first condition is open in flat families in $\underline{\mathrm{S}}$, as E is concentrated in degrees -1 and 0. As $\mathbf{D}(\mathrm{E})\in\mathrm{D}^{[-1,1]}(\mathcal{X})$ by Lemma 4.2.10, openness in $\underline{\mathrm{S}}$ of the second condition follows similarly. Now let $\mathrm{E}\in\underline{\mathrm{S}}(\mathrm{T})$ be a family of objects over some base T such that $\mathrm{rk}(\mathrm{E}_t)=-1$ and the conditions (i) and (ii) are satisfied for all closed points $t\in\mathrm{T}$. We have to show that the set $\mathrm{U}=\{t\in\mathrm{T}\mid\mathrm{E}_t\text{ satisfies (iii)}\}\subset\mathrm{T}$ is open.

Applying the dualising functor to the torsion triple in equation (4.2.21) yields the torsion triple

$$\mathbf{D}(\mathrm{Coh}_{\leq 1}(\mathcal{X}))=\langle\mathbf{D}(\mathtt{F}),\mathbf{D}(\mathtt{T}_1),\mathrm{Coh}_0(\mathcal{X})[-1]\rangle. \tag{4.2.22}$$

In particular, collapsing $\langle\mathbf{D}(\mathtt{F}),\mathbf{D}(\mathtt{T}_1)\rangle=\mathrm{Coh}_1(\mathcal{X})$ yields a torsion pair. By tilting a this torsion pair, we obtain the torsion triple

$$\mathrm{Coh}_{\leq 1}(\mathcal{X})=\langle\mathrm{Coh}_0(\mathcal{X}),\mathbf{D}(\mathtt{F}),\mathbf{D}(\mathtt{T}_1)\rangle. \tag{4.2.23}$$

Hence, $\mathtt{G}$ is closed under extensions and quotients. Since $\langle\mathtt{G},\mathbf{D}(\mathtt{T}_1)\rangle$ is an open torsion pair by assumption, a final application of Lemma 4.2.8 completes the proof. □

For later convenience, we record a particular corollary of the previous result.

Corollary 4.2.13. Let $(\mathtt{T},\mathtt{W},\tilde{\mathtt{F}})$ be an open torsion triple on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$, and assume that $\mathrm{Coh}_0(\mathcal{X})\subset\mathtt{T}$, so $\mathtt{W},\tilde{\mathtt{F}}\subset\mathrm{Coh}_1(\mathcal{X})$. The substack $\underline{\mathtt{Pair}}(\mathtt{T},\tilde{\mathtt{F}})\subset\mathfrak{Mum}_{\mathcal{X}}$ parametrising $(\mathtt{T},\tilde{\mathtt{F}})$-pairs is open. In particular, it is an algebraic stack locally of finite type.

Proof. This follows directly from the previous Proposition. □

Definition 4.2.14. Let $\alpha\in\mathrm{N}(\mathcal{X})$. We define the (open) substack

$$\underline{\mathtt{Pair}}(\mathtt{T},\mathtt{F})_\alpha=\underline{\mathtt{Pair}}(\mathtt{T},\mathtt{F})\cap\mathfrak{Mum}_{\mathcal{X},\alpha}$$

using the decomposition of $\mathfrak{Mum}_{\mathcal{X}}$ in (4.2.13).

Remark 4.2.15. By the previous result, $\underline{\mathtt{Pair}}(\mathtt{T},\mathtt{F})_\alpha$ is an algebraic stack and locally of finite type.

Remark 4.2.16. We believe that the substack $\underline{\mathtt{A}} \subset \underline{\mathrm{Coh}}^{\flat}_{\mathcal{X}}$ defines an open substack as well. In particular, this would allow one to prove the existence of an algebraic stack of higher rank pairs, obtained by replacing the condition $\mathrm{rk(E)} = -1$ by $\mathrm{rk(E)} = -r$ for $r \in \mathbf{Z}_{\geq 2}$, or of other moduli stacks of objects in $\mathtt{A}$.

The openness of $\underline{\mathtt{A}}$ in $\underline{\mathrm{Coh}}^{\flat}_{\mathcal{X}}$ should essentially follow because $\underline{\mathrm{Coh}}_{\leq 1,\mathcal{X}}$ is an open substack and because $\mathcal{O}_{\mathcal{X}}$ is a rigid sheaf in that its tangent space

$$\mathrm{Ext}^1(\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}) = \mathrm{H}^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = 0 \tag{4.2.24}$$

in the moduli space of (stable) sheaves. However, we did not manage to find an elegant proof of this belief. Therefore, we have chosen to construct the stacks $\underline{\mathtt{Pair}}(\mathtt{T},\mathtt{F})$ indirectly as substacks of $\underline{\mathtt{A}}$, by passing through the open substack $\underline{\mathtt{S}} \subset \underline{\mathrm{Coh}}^{\flat}_{\mathcal{X}} \subset \mathfrak{Mum}_{\mathcal{X}}$.

4.2.3 Gerbe structure and counting invariants

Donaldson–Thomas and Pandharipande–Thomas invariants count a particular type of *simple* objects in $\mathrm{D}(\mathcal{X})$, i.e., objects $\mathrm{E} \in \mathrm{D}(\mathcal{X})$ such that $\mathrm{Aut(E)} = \mathbf{C}^{\times}$. They fit in a bigger picture, recently advanced by D. Piyaratne and Y. Toda in [PT16, §5], counting (semi)stable objects in the bounded derived category of a Calabi–Yau threefold with respect to a Bridgeland stability condition [Bri07].

Let $(\mathtt{T},\mathtt{F})$ be a torsion pair on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$. For $(\mathtt{T},\mathtt{F})$-pairs to arise as stable objects of a hypothetical Bridgeland stability condition on $\mathrm{D}(\mathcal{X})$, they should at least be simple in the above sense. Pairs defined by a *numerical* torsion pair satisfy this property.

Definition 4.2.17. Let $(\mathtt{T},\mathtt{F})$ be a torsion pair on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$. It is said to be *numerical* if $[\mathrm{T}] = [\mathrm{F}]$ in $\mathrm{N}(\mathcal{X})$ for $\mathrm{T} \in \mathtt{T}$ and $\mathrm{F} \in \mathtt{F}$, implies that $\mathrm{T} = 0 = \mathrm{F}$.

Example 4.2.18. Any torsion pair induced by a slope function on $\mathrm{N}_{\leq 1}(\mathcal{X})$, as described in Remark 4.1.11, is numerical. Indeed, objects with the same class in $\mathrm{N}_{\leq 1}(\mathcal{X})$ have the same slope. Most, if not all, of the pairs we consider are induced by a slope function.

Example 4.2.19. Not all torsion pair are numerical, however. For a non-example, let $\mathtt{T}_x$ be the subcategory $\mathrm{Coh}_x(\mathcal{X})$ of sheaves supported at a non-stacky point $x \in \mathcal{X}$. By Lemma 2.1.17 this defines a torsion pair on $\mathrm{Coh}(\mathcal{X})$. But $[\mathcal{O}_x] = [\mathcal{O}_y]$ in $\mathrm{N}(\mathcal{X})$ also holds for non-stacky $x \neq y \in \mathcal{X}$, hence the torsion pair induced by $\mathtt{T}_x$ is not numerical.

Pairs defined by a numerical torsion pair are simple objects in $\mathrm{D}(\mathcal{X})$.

Lemma 4.2.20. Let $(\mathtt{T},\mathtt{F})$ be a numerical torsion pair on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$. Let E be a $(\mathtt{T},\mathtt{F})$-pair in the sense of Definition 4.1.14. Then $\mathrm{Aut(E)} = \mathbf{C}^{*}$.

Proof. Let $\phi\colon \mathrm{E} \to \mathrm{E}$ be an endomorphism of $\mathrm{E} \in \mathtt{Pair}(\mathtt{T},\mathtt{F})$. If $\operatorname{im}\phi$ has rank 0, then by definition we must have $\operatorname{im}\phi \in \mathtt{F} \cap \mathtt{T} = 0$. If $\operatorname{im}\phi$ has rank -1, then $\ker\phi \in \mathtt{F}$, and $\operatorname{coker}\phi \in \mathtt{T}$. But since $[\operatorname{coker}\phi] = [\ker\phi]$ in $\mathrm{N}(\mathcal{X})$, we have $\ker\phi = \operatorname{coker}\phi = 0$. Thus every endomorphism of E is an automorphism, and it follows that $\mathrm{Aut}(\mathrm{E}) = \mathbf{C}^*$. □

Consider the heart of a bounded t-structure $\mathtt{C} = \mathrm{Coh}^\flat(\mathcal{X}) \subset \mathrm{D}(\mathcal{X})$, introduced in Definition 4.1.2, with moduli stack $\underline{\mathtt{C}} \subset \mathfrak{Mum}_{\mathcal{X}}$. Recall the graded Hall algebra $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$ from 2.3.28, and recall that the moduli stack of $(\mathtt{T},\mathtt{F})$-pairs is an algebraic stack locally of finite type. Under suitable finiteness assumptions, this means that the moduli stack $\underline{\mathtt{Pair}}(\mathtt{T},\mathtt{F})$ defines an element in the *regular* graded algebra $\mathrm{H}_{\mathrm{gr,reg}}(\mathtt{C})$

Corollary 4.2.21. Let $(\mathtt{T},\mathtt{F})$ be an open numerical torsion pair on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$. Assume that $\underline{\mathtt{Pair}}(\mathtt{T},\mathtt{F})_\alpha$ is of finite type for every $\alpha \in \mathrm{N}(\mathtt{A}) = \mathbf{Z} \oplus \mathrm{N}_{\leq 1}(\mathcal{X})$.

1. $\mathcal{P}(\mathtt{T},\mathtt{F}) \coloneqq [\underline{\mathtt{Pair}}(\mathtt{T},\mathtt{F}) \subset \underline{\mathtt{C}}]$ defines an element in $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$.

2. Moreover, $(\mathbf{L}-1)\mathcal{P}(\mathtt{T},\mathtt{F}) \in \mathrm{H}_{\mathrm{gr,reg}}(\mathtt{C})$ is graded-regular.

Proof. The moduli stack $\underline{\mathtt{Pair}}(\mathtt{T},\mathtt{F})$ exists by an application of Proposition 4.2.12. The first claim then follows by our assumption and by what it means to define an element in $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$ as per Definition 2.3.30. As for the second claim, by [ACV03, Thm. 5.1.5], the moduli stack of pairs $\underline{\mathtt{Pair}}(\mathtt{T},\mathtt{F})_\alpha$ has a coarse space, over which it is a $\mathbf{C}^*$-gerbe by the previous Lemma. In general, this coarse space is only an algebraic space. By the comparison isomorphism of Lemma 2.3.6, the result now follows. □

We are now in a position to define $(\mathtt{T},\mathtt{F})$-pair counting invariants

Definition 4.2.22. Let $(\mathtt{T},\mathtt{F})$ be an open numerical torsion pair on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$, and assume that $\underline{\mathtt{Pair}}(\mathtt{T},\mathtt{F})_\alpha$ is of finite type for all $\alpha \in \mathrm{N}(\mathtt{A})$. We define the *virtual count of* $(\mathtt{T},\mathtt{F})$*-pairs of class* α by applying the integration morphism of 2.3.26 to be

$$\mathrm{P}_\alpha(\mathtt{T},\mathtt{F}) \coloneqq \mathrm{I}((\mathbf{L}-1)\mathcal{P}(\mathtt{T},\mathtt{F})_\alpha) \in \mathbf{Z}, \tag{4.2.25}$$

where $\mathcal{P}(\mathtt{T},\mathtt{F})_\alpha \coloneqq [\underline{\mathtt{Pair}}(\mathtt{T},\mathtt{F})_\alpha \subset \underline{\mathtt{C}}] \in \mathrm{H}(\mathtt{C})$ and $\underline{\mathtt{Pair}}(\mathtt{T},\mathtt{F})_\alpha = \underline{\mathtt{Pair}}(\mathtt{T},\mathtt{F}) \cap \mathfrak{Mum}_{\mathcal{X},\alpha}$.

Remark 4.2.23. By our definition of pairs in 4.1.14, we have $\underline{\mathtt{Pair}}(\mathtt{T},\mathtt{F})_\alpha = \varnothing$ and hence $\mathrm{P}_\alpha(\mathtt{T},\mathtt{F}) = 0$, whenever $\mathrm{rk}(\alpha) \neq -1$. It is possible to define higher rank $(\mathtt{T},\mathtt{F})$-pairs, and hence obtain non-zero higher rank invariants, but we do not do so here.

Remark 4.2.24. Given the previous remark, we use the shorthand $\underline{\mathtt{Pair}}(\mathtt{T},\mathtt{F})_\alpha$ for a one-dimensional numerical class $\alpha \in \mathrm{N}_{\leq 1}(\mathcal{X})$. By this we mean the moduli stack of $(\mathtt{T},\mathtt{F})$-pairs of class $(-1,\alpha) \in \mathrm{N}(\mathtt{A}) = \mathbf{Z} \oplus \mathrm{N}_{\leq 1}(\mathcal{X})$.

4.3 Wall-crossing formula of pairs

Let $\mathcal{X}$ denote a smooth CY3 orbifold with projective coarse moduli space, and let $(\mathtt{T}, \mathtt{F})$ denote a torsion pair on $\mathrm{Coh}_{\le 1}(\mathcal{X})$. We establish a general wall-crossing formula relating different notions of $(\mathtt{T}, \mathtt{F})$-pairs. As an application, we prove the DT/PT correspondence for hard Lefschetz orbifolds.

From now on, we write $\mathtt{C} \coloneqq \mathrm{Coh}^\flat(\mathcal{X}) = \langle \mathrm{Coh}_{\ge 2}[2], \mathrm{Coh}_{\le 1}(\mathcal{X})\rangle \subset \mathrm{D}(\mathcal{X})$ as defined in 4.1.2. Note that $\mathtt{C}$ is the heart of a bounded t-structure, and that $\mathrm{Coh}_{\le 1}(\mathcal{X})$ is closed under subobjects, extensions, and quotients in $\mathtt{C}$.

4.3.1 Formal wall-crossing

To motivate the wall-crossing identity, we first present a formal identity in an infinite-type Hall algebra. We call the identity *formal* because it holds in an algebra that does not support an integration morphism. As such, it cannot be used to deduce comparison theorems for curve counting invariants. However, it can serve as intuition.

This discussion is inspired by a similar motivation in [Bri11, §4.2]. Let St_∞ denote the 2-category of algebraic stacks locally of finite type with affine geometric stabilizers.

Definition 4.3.1. Fix a stack $\mathrm{S} \in \mathrm{St}_\infty$. The *infinite-type relative Grothendieck ring* $\mathrm{L}(\mathrm{St}_\infty /\mathrm{S})$ is defined as the relative Grothendieck ring $\mathrm{K}(\mathrm{St}/\mathrm{S})$ in Definition 2.3.13 with three modifications:

1. It is generated by symbols $[\mathrm{X} \to \mathrm{S}]$ where $\mathrm{X} \in \mathrm{St}_\infty$ *locally* of finite type,
2. The disjoint union relation is removed.[1]
3. The geometric bijection relation only holds for S-morphisms of finite type.

It is a $\mathrm{K}(\mathrm{St}/\mathbf{C})$-module via the product, as before.

Definition 4.3.2. The *infinite-type Hall algebra* of $\mathtt{C}$ is $\mathrm{H}_\infty(\mathtt{C}) \coloneqq \mathrm{L}(\mathrm{St}_\infty /\underline{\mathtt{C}})$, equipped with the product as in equation (2.3.11) and unit $\mathbf{1}_0 = [\underline{\mathtt{C}}_0 \subset \underline{\mathtt{C}}]$.

Remark 4.3.3. As before, $\mathrm{H}_\infty(\mathtt{C})$ is a unital associative $\mathrm{N}(\mathcal{X})$-graded algebra. Note that $\mathrm{H}_{\mathrm{gr}}(\mathtt{C}) \subset \mathrm{H}_\infty(\mathtt{C})$ is naturally a graded subalgebra where the homogeneous parts X_α of the symbols $[\mathrm{X} \to \underline{\mathtt{C}}]$ are of finite type for all $\alpha \in \mathrm{N}(\mathcal{X})$.

We introduce the elements of $\mathrm{H}_\infty(\mathtt{C})$ that appear in the formal wall-crossing formula. The infinite-type Hall algebra contains the element $\mathbf{1}_{\mathtt{C}} = [\underline{\mathtt{C}} \to \underline{\mathtt{C}}]$ corresponding to the whole category $\mathtt{C}$. Note that $\mathbf{1}_{\mathtt{C}}$ does not define an element in $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$ because the

[1] This is necessary for otherwise the algebra would be trivial. Indeed, any element $\mathrm{X} \in \mathrm{L}(\mathrm{St}_\infty /\mathrm{S})$ satisfies $\bigsqcup_{n \in \mathbf{N}} \mathrm{X} = (\bigsqcup_{n\in\mathbf{N}} \mathrm{X}) \sqcup \mathrm{X}$, so by the disjoint union relation it would follow that $\mathrm{X} = 0$.

substacks $\underline{\mathtt{C}}_\alpha$ are in general *not* of finite type for $\alpha \in \mathrm{N}(\mathcal{X})$; indeed, this is precisely where a stability condition generally comes in.

Given a subcategory $\mathtt{B} \subset \mathtt{C}$ admitting an open[2] moduli stack $\underline{\mathtt{B}} \subset \underline{\mathtt{C}}$, there is an analogous element $\mathbf{1}_{\mathtt{B}} = [\underline{\mathtt{B}} \subset \underline{\mathtt{C}}]$ in $\mathrm{H}_\infty(\mathtt{C})$. As an exception, to ease notation, we write $\mathbf{1}_{\leq 1}(\mathcal{X}) = [\underline{\mathrm{Coh}}_{\leq 1,\mathcal{X}} \subset \underline{\mathtt{C}}]$.

Lemma 4.3.4. Let $(\mathtt{T},\mathtt{F})$ be an open torsion pair on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$. There is an identity $\mathbf{1}_{\mathtt{T}} * \mathbf{1}_{\mathtt{F}} = \mathbf{1}_{\leq 1}(\mathcal{X})$ in $\mathrm{H}_\infty(\mathtt{C})$

Proof. This is [Bri11, Lem. 4.1]. We repeat the proof here as its argument also proves the formal wall-crossing identity.

Consider the diagram (2.3.11) representing the product $\mathbf{1}_{\mathtt{T}} * \mathbf{1}_{\mathtt{F}} = [\underline{\mathtt{T}} * \underline{\mathtt{F}} \xrightarrow{\pi_2 \circ a} \underline{\mathtt{C}}]$:

$$\begin{tikzcd} \underline{\mathtt{T}} * \underline{\mathtt{F}} \arrow[r, "a"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{Ex} \arrow[r, "\pi_2"] \arrow[d, "(\pi_1,\pi_3)"] & \underline{\mathtt{C}} \\ \underline{\mathtt{T}} \times \underline{\mathtt{F}} \arrow[r] & \underline{\mathtt{C}} \times \underline{\mathtt{C}} & \end{tikzcd} \tag{4.3.1}$$

Note that an extension of an object in $\mathtt{F}$ by an object in $\mathtt{T}$ yields an object in $\mathrm{Coh}_{\leq 1}(\mathcal{X})$. As a consequence, the morphism $\pi_2 \circ a\colon \underline{\mathtt{T}} * \underline{\mathtt{F}} \to \underline{\mathtt{C}}$ factors through the open inclusion $\underline{\mathrm{Coh}}_{\leq 1,\mathcal{X}} \subset \underline{\mathtt{C}}$. We claim that the induced morphism $b\colon \underline{\mathtt{T}} * \underline{\mathtt{F}} \to \underline{\mathrm{Coh}}_{\leq 1,\mathcal{X}}$ is a geometric bijection. Indeed, by the torsion pair property, any object of $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ uniquely arises as an extension of an object in $\mathtt{F}$ by an object in $\mathtt{T}$. In other words, the induced morphism

$$b(\mathbf{C})\colon (\underline{\mathtt{T}} * \underline{\mathtt{F}})(\mathbf{C}) \to \underline{\mathrm{Coh}}_{\leq 1,\mathcal{X}}(\mathbf{C}) \tag{4.3.2}$$

on $\mathbf{C}$-valued points is an equivalence. This proves that b is a geometric bijection. Since b is a morphism of $\underline{\mathtt{C}}$-stacks, we deduce that $\mathbf{1}_{\mathtt{T}} * \mathbf{1}_{\mathtt{F}} = \mathbf{1}_{\leq 1}(\mathcal{X})$ in $\mathrm{H}_\infty(\mathtt{C})$ as claimed. ☐

By Proposition 4.1.24, a torsion pair on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ determines a torsion triple $\langle \mathtt{T},\mathtt{V},\mathtt{F}\rangle$ on $\mathtt{A}$, where the category $\mathtt{V} = \mathtt{V}(\mathtt{T},\mathtt{F})$ is defined in (4.1.18). Therefore, if we prove that $\underline{\mathtt{A}} \subset \underline{\mathtt{C}}$ is open, we obtain an identity $\mathbf{1}_{\mathtt{A}} = \mathbf{1}_{\mathtt{T}} * \mathbf{1}_{\mathtt{V}} * \mathbf{1}_{\mathtt{F}}$ in $\mathrm{H}_\infty(\mathtt{C})$ by a similar argument; we have not proven that $\underline{\mathtt{A}} \subset \underline{\mathtt{C}}$ is open, but see Remark 4.2.16.

Let $(\mathtt{T},\mathtt{F})$ be an open torsion pair on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$. Recall that we write

$$\mathcal{P}(\mathtt{T},\mathtt{F}) = [\underline{\mathtt{Pair}}(\mathtt{T},\mathtt{F}) \subset \underline{\mathtt{C}}] \in \mathrm{H}_\infty(\mathtt{C}) \tag{4.3.3}$$

for the element representing the moduli stack of $(\mathtt{T},\mathtt{F})$-pairs in the infinite-type Hall algebra. Finally, let $\mathtt{R} \subset \mathtt{A}$ denote the subcategory of objects of rank -1. Since $\underline{\mathtt{R}} \subset \underline{\mathtt{C}}$ is open by Lemma 4.2.7, we have an element $\mathbf{1}_{\mathtt{R}} \in \mathrm{H}_\infty(\mathtt{C})$.

[2] This assumption is made to guarantee that the moduli stack is algebraic and locally of finite type.

Lemma 4.3.5. Let $(\mathtt{T},\mathtt{F})$ be an open torsion pair on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$. There is an identity in $\mathrm{H}_\infty(\mathtt{C})$

$$\mathbf{1}_{\mathtt{R}} = \mathbf{1}_{\mathtt{T}} * \mathcal{P}(\mathtt{T},\mathtt{F}) * \mathbf{1}_{\mathtt{F}}. \tag{4.3.4}$$

Proof. This is proven via the argument of Lemma 4.3.4, using the fact that the product in the Hall algebra is associative. The geometric bijection follows by the torsion triple identity of Proposition 4.1.24. □

The key observation, originally due to M. Reineke for the finitary Hall algebra [Rei03], is that the left hand side does not depend on the chosen torsion pair.

Corollary 4.3.6. Let $(\mathtt{T}_\pm,\mathtt{F}_\pm)$ be two open torsion pairs on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$, and let $\mathcal{P}_-$ and $\mathcal{P}_+$ denote the corresponding elements in $\mathrm{H}_\infty(\mathtt{C})$. There is an identity

$$\mathbf{1}_{\mathtt{T}_-} * \mathcal{P}_- * \mathbf{1}_{\mathtt{F}_-} = \mathbf{1}_{\mathtt{T}_+} * \mathcal{P}_+ * \mathbf{1}_{\mathtt{F}_+}. \tag{4.3.5}$$

If these torsion pairs are induced by two stability conditions on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$, as described in Remark 4.1.11, this can be interpreted as a formal wall-crossing identity.

We make this interpretation more precise by identifying the 'wall'. Without loss of generality, we may assume that $\mathtt{T}_+ \subset \mathtt{T}_-$ and hence $\mathtt{F}_+ \supset \mathtt{F}_-$. The *wall* is the full subcategory $\mathtt{W} := \mathtt{T}_- \cap \mathtt{F}_+$. Using the argument of the proof of Proposition 4.1.24, we obtain a torsion triple

$$\mathrm{Coh}_{\leq 1}(\mathcal{X}) = \langle \mathtt{T}_+, \mathtt{W}, \mathtt{F}_- \rangle. \tag{4.3.6}$$

By assumption on the torsion pairs, the moduli stacks $\underline{\mathtt{T}}_+, \underline{\mathtt{W}}, \underline{\mathtt{F}}_- \subset \underline{\mathrm{Coh}}_{\leq 1,\mathcal{X}}$ are open. In this setup we have categories of pairs $\mathtt{Pair}(\mathtt{T}_-,\mathtt{F}_-)$, $\mathtt{Pair}(\mathtt{T}_+,\mathtt{F}_+)$, with corresponding Hall algebra elements $\mathcal{P}_-,\mathcal{P}_+$ in $\mathrm{H}_\infty(\mathtt{C})$. Similarly, there is a category of hybrid pairs[3] $\mathtt{Pair}(\mathtt{T}_+,\mathtt{F}_-)$, and the corresponding element $\mathcal{P}_\pm$ in the Hall algebra.

Lemma 4.3.7. Let $(\mathtt{T}_+,\mathtt{W},\mathtt{F}_-)$ be an open torsion triple on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ such that $\mathrm{Coh}_0(\mathcal{X}) \subset \mathtt{T}_+$. There is an identity in $\mathrm{H}_\infty(\mathtt{C})$

$$\mathbf{1}_{\mathtt{W}} * \mathcal{P}_- = \mathcal{P}_\pm = \mathcal{P}_+ * \mathbf{1}_{\mathtt{W}}. \tag{4.3.7}$$

Proof. The results of Corollary 4.3.6 yields a chain of identities

$$\mathbf{1}_{\mathtt{T}_-} * \mathcal{P}_- * \mathbf{1}_{\mathtt{F}_-} = \mathbf{1}_{\mathtt{T}_+} * \mathcal{P}_\pm * \mathbf{1}_{\mathtt{F}_-} = \mathbf{1}_{\mathtt{T}_+} * \mathcal{P}_+ * \mathbf{1}_{\mathtt{F}_+} \tag{4.3.8}$$

in the infinite-type Hall algebra $\mathrm{H}_\infty(\mathtt{C})$. By an argument as in Lemma 4.3.4, using the

[3] One can think of such pairs as 'strictly semistable on the wall $\mathtt{W}$'.

geometric bijection axiom of the Hall algebra, there are further identities

$$\mathbf{1}_{\mathtt{T}_-} = \mathbf{1}_{\mathtt{T}_+} * \mathbf{1}_{\mathtt{W}} \quad \text{and} \quad \mathbf{1}_{\mathtt{W}} * \mathbf{1}_{\mathtt{F}_-} = \mathbf{1}_{\mathtt{F}_+}. \tag{4.3.9}$$

Plugging these into equation (4.3.8) yields

$$\mathbf{1}_{\mathtt{T}_+} * (\mathbf{1}_{\mathtt{W}} * \mathcal{P}_-) + \mathbf{1}_{\mathtt{F}_-} = \mathbf{1}_{\mathtt{T}_+} * \mathcal{P}_{\pm} * \mathbf{1}_{\mathtt{F}_-} = \mathbf{1}_{\mathtt{T}_+} * (\mathcal{P}_+ * \mathbf{1}_{\mathtt{W}}) * \mathbf{1}_{\mathtt{F}_-}. \tag{4.3.10}$$

So if we can multiply by $\mathbf{1}_{\mathtt{T}_+}^{-1}$ on the left and by $\mathbf{1}_{\mathtt{F}_-}^{-1}$ on the right, the desired relations (4.3.7) follow. We prove the existence of these elements in $\mathrm{H}_\infty(\mathtt{C})$ as follows. Recall that $\mathrm{Coh}_{\leq 1}(\mathcal{X}) \subset \mathtt{C}$ is a full subcategory closed under subobjects, extensions, and products. Moreover, the moduli stack $\underline{\mathrm{Coh}}_{\leq 1,\mathcal{X}} \subset \underline{\mathtt{C}}$ is open and closed. We obtain an embedding

$$\mathrm{H}_\infty(\mathrm{Coh}_{\leq 1}(\mathcal{X})) \subset \mathrm{H}_\infty(\mathtt{C}) \tag{4.3.11}$$

of $\mathrm{N}(\mathcal{X})$-graded $\mathrm{K}(\mathrm{St}/\mathbf{C})$-algebras, where the α-graded part of $\mathrm{H}_\infty(\mathrm{Coh}_{\leq 1}(\mathcal{X}))$ is simply set to be zero if $\alpha \notin \mathrm{N}_{\leq 1}(\mathcal{X})$. The elements $\mathbf{1}_{\mathtt{T}_+}$ and $\mathbf{1}_{\mathtt{F}_-}$ are invertible in the subalgebra $\mathrm{H}_\infty(\mathrm{Coh}_{\leq 1}(\mathcal{X}))$, and hence in $\mathrm{H}_\infty(\mathtt{C})$. This completes the proof. □

Remark 4.3.8. The above result suggests that if the two torsion pairs are *close enough*, i.e., if the 'wall' between them is small enough to cross, i.e., if the moduli stack $\underline{\mathtt{W}}$ parametrising objects in $\mathtt{W}$ is sufficiently finite[4], it is possible to integrate this identity into the quantum torus $\mathbf{Q}[\mathrm{N}(\mathcal{X})]$. This would yield a relation between the virtual counts of $(\mathtt{T}_-,\mathtt{F}_-)$-pairs and $(\mathtt{T}_+,\mathtt{F}_+)$-pairs.

4.3.2 The numerical wall-crossing formula

We describe conditions to make the discussion of the previous section precise. Throughout this section, let $(\mathtt{T}_\pm,\mathtt{F}_\pm)$ be two open torsion pairs on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ such that $\mathrm{Coh}_0(\mathcal{X}) \subset \mathtt{T}_+ \subset \mathtt{T}_-$, let $\mathtt{W} = \mathtt{T}_- \cap \mathtt{F}_+$ be the corresponding subcategory of walls with moduli stack $\underline{\mathtt{W}}$. We let

$$\mathrm{Coh}_{\leq 1}(\mathcal{X}) = \langle \mathtt{T}_+, \mathtt{W}, \mathtt{F}_- \rangle \tag{4.3.12}$$

denote the induced torsion triple. Moreover, we write $\mathcal{P}_- = [\underline{\mathtt{Pair}}(\mathtt{T}_-,\mathtt{F}_-) \subset \underline{\mathtt{C}}] \in \mathrm{H}_\infty(\mathtt{C})$ and similarly for $\mathcal{P}_+$. Finally, we write $\mathcal{P}_\pm$ for the element in the infinite-type Hall algebra associated to $\underline{\mathtt{Pair}}(\mathtt{T}_+,\mathtt{F}_-)$.

First, we provide sufficient conditions for equation (4.3.7) to hold in $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$.

[4]The precise condition is that $\mathtt{W}$ be *decompositionally finite*, see Definition 4.3.10.

Lemma 4.3.9. In the above setup, assume that $\mathcal{P}_\pm$ defines an element in the graded Hall algebra $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$. Then we have an identity in algebra $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$

$$\mathbf{1}_{\mathtt{W}} * \mathcal{P}_- = \mathcal{P}_\pm = \mathcal{P}_+ * \mathbf{1}_{\mathtt{W}} \tag{4.3.13}$$

and $\mathcal{P}_-, \mathcal{P}_+$, and $\mathbf{1}_{\mathtt{W}}$ define elements in $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$ as well.

Proof. By Lemma 4.3.7, the above identity is valid in $\mathrm{H}_\infty(\mathtt{C})$. It is proven by constructing finite type morphisms of algebraic stacks (in fact, open immersions)

$$\begin{aligned} \underline{\mathtt{W}} * \underline{\mathtt{Pair}}(\mathtt{T}_-, \mathtt{F}_-) &\to \underline{\mathtt{Pair}}(\mathtt{T}_+, \mathtt{F}_-) \\ \underline{\mathtt{Pair}}(\mathtt{T}_+, \mathtt{F}_+) * \underline{\mathtt{W}} &\to \underline{\mathtt{Pair}}(\mathtt{T}_+, \mathtt{F}_-) \end{aligned} \tag{4.3.14}$$

that are geometric bijections. By assumption, $\underline{\mathtt{Pair}}(\mathtt{T}_+, \mathtt{F}_-)_\alpha$ is of finite type for every $\alpha \in \mathrm{N}(\mathcal{X})$. By the open immersions $\underline{\mathtt{W}}, \underline{\mathtt{Pair}}(\mathtt{T}_-, \mathtt{F}_-), \underline{\mathtt{Pair}}(\mathtt{T}_+, \mathtt{F}_+) \subset \underline{\mathtt{Pair}}(\mathtt{T}_-, \mathtt{F}_+)$ it follows that $\mathcal{P}_-, \mathcal{P}_+$, and $\mathbf{1}_{\mathtt{W}}$ also define elements in $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$. We conclude that equation (4.3.13) holds in $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$ as claimed. □

Second, we impose a first 'smallness' condition on the full subcategory of walls $\mathtt{W}$, which we think of as the two torsion pairs being sufficiently 'close'. This condition allows us to apply the integration morphism to equation (4.3.13) and obtain a formula in the (formal) quantum torus $\mathbf{Q}\{\mathrm{N}(\mathcal{X})\}$ relating the virtual counts of pairs.

Definition 4.3.10. A full subcategory $\mathtt{W} \subset \mathrm{Coh}_{\leq 1}(\mathcal{X})$ is *decompositionally finite* if

1. $\mathtt{W}$ is closed under direct sums and summands;
2. $\mathtt{W}$ defines an element of $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$, in the sense of Definition 2.3.30;
3. if $\alpha \in \mathrm{N}_{\leq 1}(\mathcal{X})$, there are only finitely many ways of writing $\alpha = \alpha_1 + \cdots + \alpha_n$, with each α_i the class of a non-zero element in $\mathtt{W}$, i.e., $\mathtt{W}$ is a finite length category.

We arrive at D. Joyce's fundamental No-Poles Theorem. It allows us to formally take the logarithm of the element $\mathbf{1}_{\mathtt{W}}$ in the Hall algebra. Moreover, roughly speaking, it states that the corresponding element can be represented by an (infinite) sum of elements represented by quotient stacks of the form $[\mathrm{Y}/\mathbf{C}^\times]$ where Y is a variety; in other words, $(\mathbf{L} - 1) = [\mathbf{C}^\times]$ times this logarithm is regular and, as such, has 'no poles'.

Theorem 4.3.11. If $\mathtt{W}$ is decompositionally finite, then

$$(\mathbf{L} - 1) \log(\mathbf{1}_{\mathtt{W}}) \in \mathrm{H}_{\mathrm{gr,reg}}(\mathtt{C}).$$

Proof. See [Joy07, Thm. 8.7], or also[Bri11, Thm. 6.3]. A conceptually clearer approach is given in [BR16], but the CY3 case is left to the reader; see the forthcoming [BCR]. □

And third, we provide a second 'smallness' condition to ensure that the walls interact with pairs on either side in a finite type manner, i.e., in such a way as to ensure that any wall-crossing is well-defined.

Definition 4.3.12. Let $\langle \mathtt{T}_+, \mathtt{W}, \mathtt{F}_- \rangle$ be an open torsion triple on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$. We say that it is *wall-crossing material* if

1. $\mathtt{W}$ is decompositionally finite,
2. the category of hybrid pairs $\mathtt{Pair}(\mathtt{T}_+, \mathtt{F}_-)$ defines an element of $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$, and
3. for every class $\alpha \in \mathrm{N}_{\leq 1}(\mathcal{X})$ such that $\underline{\mathtt{Pair}}(\mathtt{T}_+, \mathtt{F}_+)_\alpha \neq \varnothing$, there are at most finitely many ways of writing $\alpha = \alpha' + \alpha''$ with
$$\underline{\mathtt{W}}_{\alpha'} \neq \varnothing \neq \underline{\mathtt{Pair}}(\mathtt{T}_-, \mathtt{F}_-)_{\alpha''}.$$

Remark 4.3.13. By Lemma 4.3.9, the second condition implies that the subcategories $\mathtt{Pair}(\mathtt{T}_-, \mathtt{F}_-), \mathtt{Pair}(\mathrm{T}_+, \mathtt{F}_+)$, and $\mathtt{W}$ also define elements of the graded Hall algebra.

Example 4.3.14. The first condition holds when $\mathtt{W} = \mathrm{Coh}_0(\mathcal{X})$, i.e., in the setting of the DT/PT correspondence to be discussed below. The third part of this condition follows from Lemma 2.1.38, or from the fact that $\mathrm{Coh}_0(\mathcal{X})$ is noetherian and artinian.

The second condition holds for example when $\underline{\mathtt{T}}_+, \underline{\mathtt{W}}$, and $\underline{\mathtt{F}}_-$ are open in $\underline{\mathrm{Coh}}_{\leq 1, \mathcal{X}}$, provided that the stack $\underline{\mathtt{Pair}}(\mathtt{T}_+, \mathtt{F}_-)_\alpha$ is of finite type for all $\alpha \in \mathrm{N}_{\leq 1}(\mathcal{X})$.

Recall that the product and Poisson bracket in the formal quantum torus $\mathbf{Q}\{\mathrm{N}(\mathcal{X})\}$ are not well defined for all elements. The third condition in the above definition ensures that the right hand side of equation (4.3.15) in the next Theorem is well defined.

We now give conditions for the wall-crossing formula of equation (4.3.13) to integrate to a numerical finite-type wall-crossing formula in the (formal) quantum torus.

Definition 4.3.15. We call a torsion triple $(\mathtt{T}_+, \mathtt{W}, \mathtt{F}_-)$ on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ *numerical* if the two induced torsion pairs $(\mathtt{T}_+, \mathtt{F}_+)$ and $(\mathtt{T}_-, \mathtt{F}_-)$ are numerical in the sense of Definition 4.2.17, where $\mathtt{F}_+ = \langle \mathtt{W}, \mathtt{F}_- \rangle$ and $\mathtt{T}_- = \langle \mathtt{T}_+, \mathtt{W} \rangle$ denote the extension-closures.

Theorem 4.3.16. Let $(\mathtt{T}_+, \mathtt{W}, \mathtt{F}_-)$ be an open numerical torsion triple on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$, and assume that it is wall-crossing material. Then $w \coloneqq \mathrm{I}_{\mathrm{gr}}\left((\mathbf{L} - 1) \log \mathbf{1}_{\mathtt{W}}\right)$ is well defined in $\mathbf{Q}\{\mathrm{N}(\mathcal{X})\}$, and there is the identity
$$\mathrm{I}_{\mathrm{gr}}\left((\mathbf{L} - 1)\mathcal{P}_+\right) = \exp\left(\{w, -\}\right) \mathrm{I}_{\mathrm{gr}}\left((\mathbf{L} - 1)\mathcal{P}_-\right) \tag{4.3.15}$$
in $\mathbf{Q}\{\mathrm{N}(\mathcal{X})\}$, where $\mathcal{P}_\pm = [\underline{\mathtt{Pair}}(\mathtt{T}_\pm, \mathtt{F}_\pm) \subset \underline{\mathtt{C}}] \in \mathrm{H}_{\mathrm{gr}}(\mathtt{C})$ as before.

Proof. The result follows by the arguments of [Bri11, Cor. 6.4] and equation (4.3.13):

$$\begin{aligned}\mathcal{P}_+ &= \mathbf{1}_{\mathtt{W}} * \mathcal{P}_- * \mathbf{1}_{\mathtt{W}}^{-1} \\ &= \exp(\log(\mathbf{1}_{\mathtt{W}})) * \mathcal{P}_- * \exp(-\log(\mathbf{1}_{\mathtt{W}})) \\ &= \exp(\mathrm{Ad}(\log(\mathbf{1}_{\mathtt{W}}))) \circ \mathcal{P}_-\end{aligned} \tag{4.3.16}$$

using the Baker–Campbell–Hausdorff formula. Here $\mathrm{Ad}(a) \circ x := a * x - x * a$ denotes the adjoint action of $a \in \mathrm{H}_{\mathrm{gr}}(\mathrm{Coh}_{\leq 1}(\mathcal{X}))$ on $x \in \mathrm{H}_{\mathrm{gr}}(\mathtt{C})$ whenever it is well defined. Using the Poisson bracket, this can be written as

$$\mathrm{Ad}(\log(\mathbf{1}_{\mathtt{W}})) \circ x = (\mathbf{L}-1)\{\log(\mathbf{1}_{\mathtt{W}}), x\} = \{(\mathbf{L}-1)\log(\mathbf{1}_{\mathtt{W}}), x\} \tag{4.3.17}$$

where $(\mathbf{L}-1)\log(\mathbf{1}_{\mathtt{W}}) \in \mathrm{H}_{\mathrm{gr,reg}}(\mathtt{C})$ by Theorem 4.3.11 since $\mathtt{W}$ is decompositionally finite. By Corollary 4.2.21, we also have $(\mathbf{L}-1)\mathcal{P}_\pm \in \mathrm{H}_{\mathrm{gr,reg}}(\mathtt{C})$ as well. Multiplying both sides by $(\mathbf{L}-1)$ and projecting to $\mathrm{H}_{\mathrm{gr,sc}}(\mathtt{C})$ yields

$$(\mathbf{L}-1)\mathcal{P}_+ = \exp((\mathbf{L}-1)\log(\mathbf{1}_{\mathtt{W}})) \circ (\mathbf{L}-1)\mathcal{P}_-. \tag{4.3.18}$$

Note that the product on the right hand side of equation (4.3.15) is well-defined by the fact that the torsion triple $\mathtt{W}$ is wall-crossing material. Applying the integration morphism I_{gr} of 2.3.26 concludes the proof. □

4.3.3 Orbifold DT/PT correspondence

As a first application of the wall-crossing formula, we prove the orbifold DT/PT correspondence. Thus, we assume that $\mathcal{X}$ is a smooth CY3 orbifold that satisfies the hard Lefschetz condition of Definition 2.4.13 and has a projective coarse moduli space X. Fix an ample line bundle A on X and a self-dual generating vector bundle V on $\mathcal{X}$; the latter exists by Lemma 2.1.30. Recall the modified Hilbert polynomial $p_{\mathrm{F}}(k) = \chi(\mathrm{V}, \mathrm{F} \otimes \mathrm{A}^{\otimes k})$ of a sheaf $\mathrm{F} \in \mathrm{Coh}_{\leq 1}(\mathcal{X})$, and the degree $\deg(\mathrm{F}) := p_{\mathrm{F}}(0)$ from Definition 2.1.31.

The proof we give is very similar in spirit to T. Bridgeland's and Y. Toda's original proofs of the DT/PT correspondence for CY3 varieties, given in [Bri11, Tod10a, Tod16a], as it follows their strategy. However, we emphasize that pairs are two-term complexes as opposed to sheaves with a section; see Remark 4.1.21.

First, we collect the necessary ingredients to state the correspondence and verify that we may apply the numerical finite-type wall-crossing formula of Theorem 4.3.16.

Lemma 4.3.17. Let $\mathtt{T}_{\mathrm{DT}} = 0, \mathtt{W} = \mathrm{Coh}_0(\mathcal{X})$, and $\mathtt{F}_{\mathrm{PT}} = \mathrm{Coh}_1(\mathcal{X})$. Then $(\mathtt{T}_{\mathrm{DT}}, \mathtt{W}, \mathtt{F}_{\mathrm{PT}})$ is an open numerical torsion triple on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ that is wall-crossing material.

Proof. Clearly $\mathtt{W}$ is decompositionally finite. Recall that $\mathtt{F}_{\mathrm{DT}} = \langle \mathtt{W}, \mathtt{F}_{\mathrm{PT}} \rangle = \mathrm{Coh}_{\leq 1}(\mathcal{X})$ and $\mathtt{T}_{\mathrm{PT}} = \langle \mathtt{T}_{\mathrm{DT}}, \mathtt{W} \rangle = \mathrm{Coh}_0(\mathcal{X})$. All corresponding moduli stacks are open substacks of $\underline{\mathrm{Coh}}_{\leq 1, \mathcal{X}}$, and hence of $\underline{\mathtt{C}}$. It follows by Proposition 4.2.12 that $\underline{\mathtt{P}}_{\mathrm{PT}} := \underline{\mathtt{Pair}}(\mathtt{T}_{\mathrm{PT}}, \mathtt{F}_{\mathrm{PT}})$ defines an open substack of $\underline{\mathtt{C}}$. This proposition does not apply to the other two stacks of pairs, because the property $\mathrm{Coh}_0(\mathcal{X}) \subseteq \mathtt{T}_{\mathrm{DT}}$ fails. To see that $\underline{\mathtt{P}}_{\mathrm{DT}} := \underline{\mathtt{Pair}}(\mathtt{T}_{\mathrm{DT}}, \mathtt{F}_{\mathrm{DT}})$ and $\underline{\mathtt{P}}_h := \underline{\mathtt{Pair}}(\mathtt{T}_{\mathrm{DT}}, \mathtt{F}_{\mathrm{PT}})$ define open substacks of $\underline{\mathtt{C}}$ nonetheless, note that

$$\begin{aligned} \mathtt{P}_{\mathrm{DT}} &= \{\mathrm{E} \in \mathtt{A} \mid \mathrm{rk}(\mathrm{E}) = -1, \mathrm{H}^0(\mathrm{E}) = 0\}, \text{ and} \\ \mathtt{P}_h &= \{\mathrm{E} \in \mathtt{A} \mid \mathrm{rk}(\mathrm{E}) = -1, \mathrm{H}^0(\mathrm{E}) \in \mathrm{Coh}_0(\mathcal{X})\}. \end{aligned} \tag{4.3.19}$$

Thus, these claims follow from Lemma 4.2.8, since $\underline{\mathtt{S}} \subset \underline{\mathtt{C}}$ is open. Moreover, the DT and PT torsion pairs are numerical, so by Corollary 4.2.21 their moduli stacks, $\underline{\mathtt{P}}_{\mathrm{DT}}$ and $\underline{\mathtt{P}}_{\mathrm{PT}}$ respectively, are $\mathbf{C}^\times$-gerbes over their coarse moduli spaces.

Using the cohomological criterion of Proposition 4.1.16, we see that

1. $(\mathtt{T}_{\mathrm{DT}}, \mathtt{F}_{\mathrm{DT}})$-pairs are ideal sheaves of curves on $\mathcal{X}$ shifted by [1], and
2. $(\mathtt{T}_{\mathrm{PT}}, \mathtt{F}_{\mathrm{PT}})$-pairs are stable pairs on $\mathcal{X}$.

Fix a class $\alpha \in \mathrm{N}_{\leq 1}(\mathcal{X})$. The coarse space of $\underline{\mathtt{P}}_{\mathrm{DT}}(\alpha)$ is the Quot scheme in the sense of M. Olsson and J. Starr [OS03], parametrising quotients $\mathcal{O}_{\mathcal{X}} \twoheadrightarrow \mathrm{F}$ in $\mathrm{Coh}(\mathcal{X})$ such that $[\mathrm{F}] = \alpha$. By Theorem 2.1.32, the scheme

$$\mathrm{Quot}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}, p) := \bigcup_{\substack{\alpha \in \mathrm{N}(\mathcal{X}) \\ p_\alpha = p}} \mathrm{Quot}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}, \alpha) \tag{4.3.20}$$

is projective for every polynomial $p \in \mathbf{Z}[t]$. In particular, $\mathrm{Quot}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}, \alpha)$ and hence $\underline{\mathtt{P}}_{\mathrm{DT}}(\alpha)$ is of finite type. We conclude that $\mathtt{P}_{\mathrm{DT}}$ defines an element in $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$.

Note that the modified Hilbert polynomial satisfies $p_{\alpha+c}(k) = p_\alpha(k) + \deg(c)$ for all $c \in \mathrm{N}_0(\mathcal{X})$. On the one hand, we have that $\mathrm{Quot}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}, p_{\alpha+c}) = \varnothing$ for $c \in \mathrm{N}_0(\mathcal{X})$ such that $\deg(c) \ll 0$ by Lemma 5.2.8. On the other hand, $\deg(\mathrm{W}) \equiv \chi(\mathrm{V}, \mathrm{W}) = \hom(\mathrm{V}, \mathrm{W}) \geq 0$ for all $\mathrm{W} \in \mathtt{W} = \mathrm{Coh}_0(\mathcal{X})$ so $\underline{\mathtt{W}}(d) = \varnothing$ for $d < 0$. Note that the open immersion $\underline{\mathtt{P}}_{\mathrm{DT}} * \underline{\mathtt{W}} \to \underline{\mathtt{P}}_h$ is a geometric bijection of $\underline{\mathtt{C}}$-stacks, inducing the identity $\mathcal{P}_{\mathrm{DT}} * \mathbf{1}_{\mathtt{W}} = \mathcal{P}_h$ in $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$. It follows that

$$\underline{\mathtt{P}}_h(p_\alpha) \subset \bigcup_{d \geq 0} \underline{\mathtt{P}}_{\mathrm{DT}}(p_\alpha - d) \times \underline{\mathtt{W}}(d) \tag{4.3.21}$$

is contained in a finite union of finite type stacks. As a consequence, $\underline{\mathtt{P}}_h(p_\alpha)$ and hence $\underline{\mathtt{P}}_h(\alpha)$ is of finite type for all $\alpha \in \mathrm{N}_{\leq 1}(\mathcal{X})$. Thus $\mathtt{P}_h$ defines an element in $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$ and, hence, the same follows for $\mathtt{P}_{\mathrm{PT}}$.

Finally, fix a (split) class $\alpha = (\beta, c) \in \mathrm{N}_1(\mathcal{X}) \oplus \mathrm{N}_0(\mathcal{X})$ such that $\underline{\mathrm{P}}_{\mathrm{PT}}(\beta, c) \neq \varnothing$. Given that $\mathtt{W} = \mathrm{Coh}_0(\mathcal{X})$, it remains to show that there are at most finitely many ways of writing $c = c' + c''$ in $\mathrm{N}_0(\mathcal{X})$ with $\underline{\mathtt{W}}(c') \neq \varnothing \neq \underline{\mathrm{P}}_{\mathrm{DT}}(\beta, c'')$. Since the latter is a $\mathbf{C}^\times$-gerbe over $\mathrm{Quot}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}, (\beta, c''))$, we may equivalently reason with the Quot scheme.

On the one hand, the previous argument states that $\mathrm{Quot}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}, p_\alpha - d)$ is empty for $d \gg 0$, which implies that the set

$$\mathrm{D}(\alpha) = \{d \in \mathbf{Z} \mid \mathrm{Quot}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}, p_\alpha - d) \neq \varnothing \neq \underline{\mathtt{W}}(d)\} \tag{4.3.22}$$

is bounded above. On the other hand, $\underline{\mathtt{W}}(d) \neq \varnothing$ forces $d \geq 0$, so that the set $\mathrm{D}(\alpha)$ is also bounded below. Since $\mathbf{Z}$ is discrete it follows that $\mathrm{D}(\alpha)$ is finite.

The result of Theorem 2.1.32 shows that the scheme

$$\mathrm{Quot}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}, p_\alpha - d) \supseteq \bigcup_{\substack{c \in \mathrm{N}_0(\mathcal{X}) \\ \deg(c) = d}} \mathrm{Quot}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}, \alpha - c) \tag{4.3.23}$$

is projective for all $d \in \mathbf{Z}$, and hence that only finitely many of those Quot schemes are non-empty; here we have used $p_{\alpha - c}(k) = p_\alpha(k) - \deg(c)$ for $c \in \mathrm{N}_0(\mathcal{X})$. In other words,

$$\mathrm{S}_d = \{c \in \mathrm{N}_0(\mathcal{X}) \mid \mathrm{Quot}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}, \alpha - c) \neq \varnothing \text{ and } \deg(c) = d\} \tag{4.3.24}$$

is finite for all $d \in \mathbf{Z}$. Combining both results and the $\mathbf{C}^\times$-gerbe structure, shows that

$$\{c', c'' \in \mathrm{N}_0(\mathcal{X}) \mid c = c' + c'', \underline{\mathtt{W}}(c') \neq \varnothing \neq \underline{\mathrm{P}}_{\mathrm{DT}}(\alpha - c'')\} \subseteq \bigsqcup_{d \in \mathrm{D}(\alpha)} \mathrm{S}_d \tag{4.3.25}$$

is a finite set for all $\alpha \in \mathrm{N}_{\leq 1}(\mathcal{X})$, irrespective of the condition $\underline{\mathrm{P}}_{\mathrm{PT}}(\alpha) \neq \varnothing$. This proves that $(\mathtt{T}_{\mathrm{DT}}, \mathtt{W}, \mathtt{F}_{\mathrm{PT}})$ is an open numerical torsion triple that is wall-crossing material. □

For a (split) class $\alpha = (\beta, c)$ with $\beta \in \mathrm{N}_1(\mathcal{X})$ and $c \in \mathrm{N}_0(\mathcal{X})$, we write $\mathrm{PT}_{\mathcal{X}}(\beta, c)$ for the Behrend-weighted Euler characteristic of the corresponding coarse moduli space as in (1.1.23). In other words, in terms of the integration morphism of 2.3.26, we have

$$\mathrm{I}((\mathbf{L} - 1)\mathcal{P}_{\mathrm{PT}}(\beta, c)) = \mathrm{PT}_{\mathcal{X}}(\beta, c) t^{(-1,0,\beta,c)} \tag{4.3.26}$$

in the quantum torus $\mathbf{Q}[\mathrm{N}(\mathcal{X})]$. We collect these invariants in a generating function

$$\mathrm{PT}(\mathcal{X})_\beta = \sum_{c \in \mathrm{N}_0(\mathcal{X})} \mathrm{PT}_{\mathcal{X}}(\beta, c) q^c \tag{4.3.27}$$

where we write $t^\alpha = z^\beta q^c$ for the formal variables to emphasize the splitting into curve and point classes $\mathrm{N}_{\leq 1}(\mathcal{X}) = \mathrm{N}_1(\mathcal{X}) \oplus \mathrm{N}_0(\mathcal{X})$, and we write $s = t^{-[\mathcal{O}_{\mathcal{X}}]}$ for the negative of the class of the structure sheaf. Similarly, for the Donaldson–Thomas invariants we have the generating function $\mathrm{DT}(\mathcal{X})_\beta = \sum_{c \in \mathrm{N}_0(\mathcal{X})} \mathrm{DT}_{\mathcal{X}}(\beta, c) q^c$.

We now prove the orbifold DT/PT correspondence for multi-regular curves classes.

Theorem 4.3.18. Let $\mathcal{X}$ be a smooth projective CY3 orbifold satisfying the hard Lefschetz condition, and let $\beta \in \mathrm{N}_1(\mathcal{X})$ be a multi-regular curve class. Then we have

$$\mathrm{PT}(\mathcal{X})_\beta = \frac{\mathrm{DT}(\mathcal{X})_\beta}{\mathrm{DT}(\mathcal{X})_0} \tag{4.3.28}$$

as formal power series in $\mathbf{Q}[\mathrm{N}_0(\mathcal{X})]_{\mathrm{L}_{\deg}}$ where $\mathrm{L}_{\deg}\colon \mathrm{N}_0(\mathcal{X}) \to \mathbf{Z}$ sends c to $\deg(c)$.

This theorem has previously been announced by A. Bayer (unpublished).

Proof. We apply the numerical wall-crossing formula of Theorem 4.3.16 to the open numerical torsion triple $\mathtt{T}_+ = \mathtt{T}_{\mathrm{DT}} = 0$, $\mathtt{W} = \mathrm{Coh}_0(\mathcal{X})$, and $\mathtt{F}_- = \mathtt{F}_{\mathrm{PT}} = \mathrm{Coh}_1(\mathcal{X})$. The triple $(\mathtt{T}_+, \mathtt{W}, \mathtt{F}_-)$ is wall-crossing material by Lemma 4.3.17. Moreover, its proof shows that, for all $\beta \in \mathrm{N}_1(\mathcal{X})$ and $\beta = 0$, the sets

$$\begin{aligned} \mathrm{S}_{\mathrm{DT}}(\beta) &= \{c \in \mathrm{N}_0(\mathcal{X}) \mid \underline{\mathrm{P}}_{\mathrm{DT}}(\beta, c) \neq \varnothing\}, \text{ and} \\ \mathrm{S}_{\mathrm{PT}}(\beta) &= \{c \in \mathrm{N}_0(\mathcal{X}) \mid \underline{\mathrm{P}}_{\mathrm{PT}}(\beta, c) \neq \varnothing\} \end{aligned} \tag{4.3.29}$$

are $\mathrm{L}_{\deg}$-bounded in the sense of Definition 2.5.7. Thus the products and Poisson brackets appearing in the wall-crossing formula equation (4.3.13) are well defined and lie in the ring $\mathbf{Q}[\mathrm{N}_0(\mathcal{X})]_{\mathrm{L}_{\deg}}$.

We compute both sides of the wall-crossing formula. The left hand side of equation (4.3.15) yields $\mathrm{I}_{\mathrm{gr,sc}}\big((\mathbf{L}-1)\mathcal{P}_{\mathrm{DT},\beta}\big) = \mathrm{DT}(\mathcal{X})_\beta z^\beta s$ where $s = t^{-[\mathcal{O}_{\mathcal{X}}]}$. Define the element $w = \mathrm{I}((\mathbf{L}-1)\log \mathbf{1}_{\mathtt{W}})$ which lies in $\mathrm{H}_{\mathrm{gr,reg}}(\mathtt{C})$. The right hand side of the equation yields

$$\exp(\{w, -\})\mathrm{I}_{\mathrm{gr,sc}}\big((\mathbf{L}-1)\mathcal{P}_{\mathrm{PT},\beta}\big) = \exp(\{w, -\})\mathrm{PT}(\mathcal{X})_\beta z^\beta s. \tag{4.3.30}$$

To understand the action of $\exp(\{w, -\})$ on $z^\beta s$, we decompose $w = \sum_{c \in \mathrm{N}_0(\mathcal{X})} w_c q^c$ in $\mathbf{Q}[\mathrm{N}_0(\mathcal{X})]_{\mathrm{L}_{\deg}}$ where $w_c \in \mathbf{Q}$. The action of the Poisson bracket is

$$\begin{aligned} \{q^c, z^\beta s\} &= \{q^c, z^\beta\} s + z^\beta \{q^c, s\} \\ &= 0 - (-1)^{\chi(c, \mathcal{O}_{\mathcal{X}})} \chi(c, \mathcal{O}_{\mathcal{X}}) z^\beta s \\ &= (-1)^{\chi(\mathcal{X}, c)} \chi(\mathcal{X}, c) z^\beta s. \end{aligned} \tag{4.3.31}$$

Here $\{q^c, z^\beta\}$ vanishes because $\chi(c, \beta) = 0$ by multi-regularity of β and the fact that the Euler pairing of two one-dimensional sheaves on the smooth projective threefold Y vanishes. We have also used the anti-symmetry of the Euler pairing since $\mathcal{X}$ is CY3.

It follows that the action of $\exp(\{w,-\})$ on $\mathrm{PT}(\mathcal{X})_\beta z^\beta s$ is multiplication by a factor:

$$\exp(\{w,-\})\mathrm{PT}(\mathcal{X})_\beta z^\beta s = \exp\left(\sum_{c\in \mathrm{N}_0(\mathcal{X})} (-1)^{\chi(c)}\chi(c) w_c q^c\right)\mathrm{PT}(\mathcal{X})_\beta z^\beta s, \tag{4.3.32}$$

where $\chi(c) = \chi(\mathcal{X}, c)$. So for any $\beta \in \mathrm{N}_{\mathrm{mr}}(\mathcal{X})$ we have the equality $\mathrm{DT}(\mathcal{X})_\beta = \mathrm{F}\cdot\mathrm{PT}(\mathcal{X})_\beta$ in the ring $\mathbf{Q}[\mathrm{N}_0(\mathcal{X})]_{\mathrm{L}_{\mathrm{deg}}}$. By specialising to the curve class $\beta = 0$ and using the fact that $\mathrm{PT}(\mathcal{X})_0 = 1$, we infer that $\mathrm{F} = \mathrm{DT}(\mathcal{X})_0$. This completes the proof. $\square$

Remark 4.3.19. Note that the vanishing in equation (4.3.31) is the only place in the proof where the multi-regularity of the curve class β is used. On the other hand, we have used the hard Lefschetz assumption (in that $\mathrm{N}_0(\mathcal{X})$ are sent to at most one-dimensional classes by Φ) to guarantee that $\{q^c, q^{c'}\} = 0$ for all $c, c' \in \mathrm{N}_0(\mathcal{X})$.

In order to prove a DT/PT correspondence for non-multi-regular classes, the formula (4.3.28) should be altered. Indeed, the factor F will depend on certain Euler pairings with the curve class β, as the bracket $\{q^c, z^\beta\}$ need no longer vanish.

Remark 4.3.20. The wall-crossing formula may also be applied to prove comparison theorems of curve counts on a smooth projective Calabi–Yau threefold Y. Examples are the DT/PT correspondence of 1.2.9 and the BS/fPT correspondence of 1.2.16.

We emphasize that thinking of curves as pairs rather than sheaves with a section can provide new insights. For example, reproving the BS/fPT correspondence yields an exponential formula for the series $\mathrm{DT}(\mathrm{Y})_{\mathrm{exc}}$, counting curves on Y that are contracted by the crepant resolution f, in terms of D. Joyce and Y. Song's N-invariants [JS12]. Recently, the author learned that this formula was already known to Y. Toda, see [Tod13, Thm. 5.6], on whose ideas much of the current thesis rests.

Chapter 5

A proof of the crepant resolution conjecture

We prove the crepant resolution conjecture for Donaldson–Thomas invariants, originally conjectured by J. Bryan, C. Cadman, and B. Young in [BCY12], after reinterpreting it as an equality of rational functions, as mentioned in section 1.3. Our proof contains two key ingredients. The first is the theory of *pairs*, as discussed in the previous chapter, in particular their wall-crossing under a change of stability in the motivic Hall algebra, and the numerical wall-crossing formula between the associated counting invariants. The second is the insight of J. Rennemo that crossing a certain *cluster* of walls induces a re-expansion of the rational function of counting invariants, allowing us to prove our equality.

We now describe the content of the sections of this chapter. First, we give a detailed strategy of the proof and formulate precise statements at each step. Second, we prove that the generating series of stable pair invariants on a smooth projective CY3 orbifold is the expansion of a rational function. We conjecture that, as a corollary, one can deduce a symmetry of this function induced by the derived dualising functor. Third, we prove that crossing a certain infinite set of walls (a *cluster*) induces a re-expansion of the associated rational function of counting invariants. And fourth, we show that after crossing a certain number of clusters, the expansion of the rational function is the generating function of Bryan–Steinberg invariants. Together, this proves the crepant resolution conjecture for projective CY3 orbifolds.

The results in this section are joint work with J. Rennemo and J. Calabrese [BCR].

5.1 On the proof

After recalling the key ingredients of the crepant resolution conjecture, we divide its proof into four steps. At each step we give a precise description of the statements and indicate how we prove them.

5.1.1 Setting

We recall the crepant resolution conjecture as described in section 2.4. Let $\mathcal{X}$ be a smooth CY3 orbifold satisfying the hard Lefschetz condition with projective coarse moduli space $\pi\colon \mathcal{X} \to \mathrm{X}$. Results of [BKR01, CT08] provide us with a natural crepant resolution $f\colon \mathrm{Y} \to \mathrm{X}$. The situation is summarised in the following diagram.

$$\begin{tikzcd} \mathcal{X} \arrow[dr, "\pi"'] & & \mathrm{Y} \arrow[dl, "f"] \\ & \mathrm{X} & \end{tikzcd} \tag{5.1.1}$$

The McKay equivalence $\Phi\colon \mathrm{D}(\mathrm{Y}) \to \mathrm{D}(\mathcal{X})$ of [BKR01] sends $\Phi(\mathcal{O}_{\mathrm{Y}}) = \mathcal{O}_{\mathcal{X}}$ and induces an identification of the numerical Grothendieck groups of the orbifold $\mathcal{X}$ and the resolution Y, which we denote by $\phi\colon \mathrm{N}(\mathcal{X}) \to \mathrm{N}(\mathrm{Y})$.

This isomorphism does not respect the natural filtrations on $\mathrm{N}(\mathcal{X})$ and $\mathrm{N}(\mathrm{Y})$ by dimension of support. The following diagram summarises how ϕ interacts with the grading:

$$\begin{tikzcd} \mathrm{N}_0(\mathrm{Y}) \arrow[r, hook] & \mathrm{N}_{\mathrm{exc}}(\mathrm{Y}) \arrow[r, hook] \arrow[d, equal, "\phi"] & \mathrm{N}_{\leq 1}(\mathrm{Y}) \arrow[d, equal, "\phi"] & \\ & \mathrm{N}_0(\mathcal{X}) \arrow[r, hook] & \mathrm{N}_{\mathrm{mr}}(\mathcal{X}) \arrow[r, hook] & \mathrm{N}_{\leq 1}(\mathcal{X}) \end{tikzcd} \tag{5.1.2}$$

Here $\mathrm{N}_{\mathrm{exc}}(\mathrm{Y}) \coloneqq \phi^{-1}(\mathrm{N}_0(\mathcal{X}))$ are the *exceptional* classes on Y, and $\mathrm{N}_{\mathrm{mr}}(\mathcal{X}) \coloneqq \phi(\mathrm{N}_{\leq 1}(\mathrm{Y}))$ are the *multi-regular* ones. By the hard Lefschetz condition, the former are classes supported on the one-dimensional fibres of f. The latter are those classes on $\mathcal{X}$ that correspond to one-dimensional classes on Y.

Recall the quantum torus $\mathbf{Q}[\mathrm{N}(\mathcal{X})]$ with $\mathbf{Q}$-basis $\{t^\alpha \mid \alpha \in \mathrm{N}(\mathcal{X})\}$. The formal variable t^α bookkeeps the Donaldson–Thomas invariants on $\mathcal{X}$ of class α, and we use the identification $\phi\colon \mathrm{N}(\mathrm{Y}) \to \mathrm{N}(\mathcal{X})$ for those on Y. For $\beta \in \mathrm{N}_1(\mathcal{X})$ we have the series

$$\begin{aligned} \mathrm{DT}(\mathcal{X})_\beta &\coloneqq \sum_{c \in \mathrm{N}_0(\mathcal{X})} \mathrm{DT}_{\mathcal{X}}(\beta + c) t^{\beta + c} \\ \mathrm{DT}(\mathcal{X})_0 &\coloneqq \sum_{c \in \mathrm{N}_0(\mathcal{X})} \mathrm{DT}_{\mathcal{X}}(c) t^{c} \end{aligned} \tag{5.1.3}$$

of Donaldson–Thomas invariants on the orbifold $\mathcal{X}$, and the generating series

$$\begin{aligned} \mathrm{DT}(\mathrm{Y})_\beta &:= \sum_{c \in \mathrm{N}_0(\mathcal{X})} \mathrm{DT}_\mathrm{Y}(\beta + c) t^{\beta + c} \\ \mathrm{DT}(\mathrm{Y})_{\mathrm{exc}} &:= \sum_{c \in \mathrm{N}_0(\mathcal{X})} \mathrm{DT}_\mathrm{Y}(c) t^{c} \end{aligned} \tag{5.1.4}$$

of Donaldson–Thomas invariants on the resolution Y; recall that $\phi(\mathrm{N}_{\mathrm{exc}}(\mathrm{Y})) = \mathrm{N}_0(\mathcal{X})$. In general, these series live in a certain completion of the quantum torus $\mathbf{Q}[\mathrm{N}(\mathcal{X})]$, but for now we consider them in the formal quantum torus $\mathbf{Q}\{\mathrm{N}(\mathcal{X})\}$.

We prove the crepant resolution conjecture for Donaldson–Thomas invariants.

Theorem 5.1.1. Let $\mathcal{X}$ be a smooth CY3 orbifold satisfying the hard Lefschetz condition with projective coarse moduli space. For each multi-regular curve class $\beta \in \mathrm{N}_{\mathrm{mr}}(\mathcal{X})$ there exists a rational function $f_\beta(q)$, where q denotes a multi-variable $q_1, q_2, \ldots, q_r$ and the q_i are generators of $\mathbf{Q}[\mathrm{N}_0(\mathcal{X})]$ corresponding to a basis of $\mathrm{N}_0(\mathcal{X})$, such that

1. the expansion of $f_\beta(q)$ with respect to $\mathrm{L}_{\mathrm{deg}}$ is the quotient $\mathrm{DT}(\mathcal{X})_\beta/\mathrm{DT}(\mathcal{X})_0$ of formal power series,
2. the expansion of $f_\beta(q)$ with respect to another linear function $\mathrm{L}\colon \mathrm{N}_0(\mathcal{X})_{\mathbf{R}} \to \mathbf{R}$ is the quotient $\mathrm{DT}(\mathrm{Y})_\beta/\mathrm{DT}_{\mathrm{exc}}(\mathrm{Y})$ of formal power series, and
3. the poles of $f_\beta(q)$ lie on the locus $\{q^{2\beta\cdot \mathrm{A}} - 1 = 0\}$ where $\mathrm{A} \in \mathrm{Amp}(\mathrm{X})\}$.

Remark 5.1.2. Let $\mathrm{L}\colon \mathrm{N}_0(\mathcal{X}) \to \mathbf{R}$ be a linear function and let $f = g/h$ be a rational function where $g, h \in \mathbf{Q}[\mathrm{N}_0(\mathcal{X})]$. Recall Definition 2.5.9 of the completion $\mathbf{Q}[\mathrm{N}_0(\mathcal{X})]_\mathrm{L}$ of $\mathbf{Q}[\mathrm{N}_0(\mathcal{X})]$ with respect to L. Also recall that an element $f_\mathrm{L} \in \mathbf{Q}[\mathrm{N}_0(\mathcal{X})]_\mathrm{L}$ is called the expansion of the rational function f with respect to L if $f_\mathrm{L} h = g$ holds in $\mathbf{Q}[\mathrm{N}_0(\mathcal{X})]_\mathrm{L}$. Note that $\mathbf{Q}[\mathrm{N}_0(\mathcal{X})] \subset \mathbf{Q}[\mathrm{N}_0(\mathcal{X})]_\mathrm{L}$ for all non-zero L.

To complete the statement of the theorem, we should specify in which completion of $\mathbf{Q}[\mathrm{N}(\mathcal{X})]$ the quotient $\mathrm{DT}(\mathrm{Y})_\beta/\mathrm{DT}_{\mathrm{exc}}(\mathrm{Y})$ is the expansion of the rational functions $f_\beta(q)$, i.e., we should specify the function L. For the second quotient, the completion will depend on the curve class β, and it will be specified in the proof of the theorem.

5.1.2 Strategy of the proof

We describe the four steps of our proof of Theorem 5.1.1 in detail.

Step 1: Both quotient generating series appearing in the conjecture can be expressed as a series whose coefficients count geometric objects on $\mathcal{X}$ and $f\colon \mathrm{Y} \to \mathrm{X}$ respectively. This reinterpretation allows us to apply wall-crossing methods to link the two types of objects, and their corresponding counts, directly.

The quotient $\mathrm{DT}(\mathcal{X})_\beta/\mathrm{DT}(\mathcal{X})_0$ equals the generating series of stable pair invariants on $\mathcal{X}$. Indeed, by the orbifold DT/PT correspondence proven in Theorem 4.3.18, we have the equality of generating series

$$\mathrm{PT}(\mathcal{X})_\beta = \frac{\mathrm{DT}(\mathcal{X})_\beta}{\mathrm{DT}(\mathcal{X})_0} \tag{5.1.5}$$

in the completion $\mathbf{Q}[\mathrm{N}_0(\mathcal{X})]_{\mathrm{L}_{\mathrm{deg}}}$ for every multi-regular curve class $\beta \in \mathrm{N}_{\mathrm{mr}}(\mathcal{X})$.

The quotient $\mathrm{DT}(\mathrm{Y})_\beta/\mathrm{DT}_{\mathrm{exc}}(\mathrm{Y})$ equals the generating series of Bryan–Steinberg[1] invariants. Indeed, by [BS16, Thm. 6], we have the equality of generating series

$$\mathrm{PT}_f(\mathrm{Y}/\mathrm{X})_\beta = \frac{\mathrm{DT}(\mathrm{Y})_\beta}{\mathrm{DT}_{\mathrm{exc}}(\mathrm{Y})} \tag{5.1.6}$$

for every multi-regular curve class $\beta \in \mathrm{N}_{\mathrm{mr}}(\mathcal{X})$.

This step of the proof has already been completed.

Step 2: We prove the rationality of the generating series of stable pair invariants on the orbifold $\mathcal{X}$. This step is motivated by a similar result for varieties as given in Conjecture 1.2.11. To be precise, we prove the following

Theorem 5.1.3. Let $\mathcal{X}$ be a smooth CY3 orbifold with projective coarse moduli space, and let $\beta \in \mathrm{N}_{\le 1}(\mathcal{X})$ be any curve class. Then $\mathrm{PT}(\mathcal{X})_\beta$ is the expansion of a rational function $f_\beta(q)$ with respect to $\mathrm{L}_{\mathrm{deg}}$. Here $q = (q_1, q_2, \ldots, q_r)$ are generators of $\mathbf{Q}[\mathrm{N}_0(\mathcal{X})]$ corresponding to a basis of $\mathrm{N}_0(\mathcal{X})$.

Note that we neither require β to be a multi-regular curve class, nor require $\mathcal{X}$ to satisfy the hard Lefschetz condition. This is a general rationality result about the stable pair theory on CY3 orbifolds with projective coarse moduli space.

We prove this theorem using the theory of wall-crossing and pairs as developed in the previous chapter. Let $\delta \in \mathbf{R}$. A δ-pair is a $(\mathtt{T}_\delta, \mathtt{F}_\delta)$-pair in $\mathtt{A}$, where

$$\begin{aligned} \mathtt{T}_\delta &:= \left\langle \mathrm{T} \in \mathrm{Coh}_{\le 1}(\mathcal{X}) \;\middle|\; \mathrm{T} \text{ is Nironi-semistable and } \nu(\mathrm{T}) \ge \delta \right\rangle_{\mathrm{ex}} \\ \mathtt{F}_\delta &:= \left\langle \mathrm{F} \in \mathrm{Coh}_{\le 1}(\mathcal{X}) \;\middle|\; \mathrm{F} \text{ is Nironi-semistable and } \nu(\mathrm{T}) < \delta \right\rangle_{\mathrm{ex}} \end{aligned} \tag{5.1.7}$$

defines a family of open numerical torsion pairs on $\mathrm{Coh}_{\le 1}(\mathcal{X})$, and ex denotes the extension-closure. Fix a curve class $\beta \in \mathrm{N}_{\le 1}(\mathcal{X})$. The notion of δ-pair of class $\le \beta$

[1] See section 1.2.5 for a discussion of this comparison theorem, and see section 2.4.3 for details on, and examples of, Bryan–Steinberg invariants.

is locally constant outside a discrete set of *walls* $\mathrm{W}_\beta \subset \mathbf{R}$. Let $\delta \in \mathrm{W}_\beta$, and write

$$\mathtt{W}_\delta := \Big\langle \mathrm{W} \in \mathrm{Coh}_{\leq 1}(\mathcal{X}) \;\Big|\; \mathrm{W} \text{ is Nironi-semistable of } \nu(\mathrm{W}) = \delta \Big\rangle_{\mathrm{ex}} \tag{5.1.8}$$

for the category of objects 'on the wall'. For $\epsilon > 0$ small enough, the triple $(\mathtt{T}_{\delta+\epsilon}, \mathtt{W}_\delta, \mathtt{F}_{\delta-\epsilon})$ is wall-crossing material, so we may apply the numerical wall-crossing theorem 4.3.16 to relate the pair counts before and after the wall.

We prove that the generating series of ϵ-pairs for $0 < \epsilon \ll 1$ is a rational function (a Laurent polynomial), and that crossing a δ-wall preserves this property (but not the rational function). Finally, we show that δ-pairs are stable pairs for $\delta \to \infty$.

We conjecture that a similar strategy, combined with the symmetry between δ-pairs and $(-\delta)$-pairs induced by the derived dual $\mathbf{D}$, yields the following.

Conjecture 5.1.4. Let $\mathcal{X}$ be a smooth CY3 orbifold with projective coarse moduli space, and let $\beta \in \mathrm{N}_{\leq 1}(\mathcal{X})$ be a curve class. The rational function $f_\beta(q)$ satisfies

$$f_\beta(q) = f_{\beta^\vee}(q^\vee), \tag{5.1.9}$$

where $(-)^\vee \colon \mathbf{Q}[\mathrm{N}(\mathcal{X})] \to \mathbf{Q}[\mathrm{N}(\mathcal{X})]$ is the anti-isomorphism induced by $\mathbf{D}$.

Note that for multi-regular β we have $\beta^\vee = \beta + c_\beta$ for some $c_\beta \in \mathrm{N}_0(\mathcal{X})$.

Step 3: We introduce a second family of stability conditions on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ parametrised by a positive real number $\gamma \in \mathbf{R}_{>0}$. The associated γ-pairs interpolate between stable pairs for $0 < \gamma \ll 1$ and Bryan–Steinberg pairs for $\gamma \to \infty$; recognising the latter as Bryan–Steinberg pairs is non-trivial and constitutes the fourth and final step of the proof.

Fix a multi-regular curve class $\beta \in \mathrm{N}_{\mathrm{mr}}(\mathcal{X})$. The notion of γ-pair of class $\leq \beta$ is locally constant outside a discrete and *finite* set of walls $\mathrm{V}_\beta \subset \mathbf{R}_{>0}$. Fix a $\gamma \in \mathrm{V}_\beta$. Unfortunately, the corresponding torsion triple is not wall-crossing material since the wall is 'too large' to cross. Thus, the wall-crossing theorem does not apply.

We refine our stability conditions to a two-parameter family of stability conditions depending on $(\gamma, \eta) \in \mathbf{R}_{>0} \times \mathbf{R}$. The associated notion of (γ, η)-pairs only depends on η when $\gamma \in \mathrm{V}_\beta$. Varying η subdivides the γ-wall into an infinite *cluster* of η-walls. The numerical wall-crossing theorem 4.3.16 *does* apply to each η-wall.

Let $\gamma_0 \in \mathrm{V}_\beta$ denote the smallest wall, so γ-pairs for $0 < \gamma < \gamma_0$ are precisely stable pairs on $\mathcal{X}$. Their generating function is a certain expansion of the rational function $f_\beta(q)$. We prove that crossing the entire cluster of η-walls on the γ_0-wall *re-expands* this rational function as a function of $q = (q_1, q_2, \dots, q_r)$. In other words, the coefficients of this re-expanded function are the counting invariants of γ-pairs for $\gamma_0 < \gamma < \gamma_1$. Here $\gamma_1 \in \mathrm{V}_\beta$ denotes the next wall for pairs of class $\leq \beta$.

The same argument shows that the other walls $\gamma_i \in \mathrm{V}_\beta$ can be crossed at the cost of re-expanding the rational function. In particular, after crossing all walls $\gamma_0 < \gamma_1 < \dots < \gamma_s$ in V_β, we deduce that the generating series of counts of γ-pairs of class $\leq \beta$ for $\gamma > \gamma_s$ is a certain re-expansion of the rational function $f_\beta(q)$.

Step 4: Let β be a multi-regular curve class as above. We show that after crossing all walls $\gamma_0 < \gamma_1 < \dots < \gamma_s$ in V_β, the γ-pairs of class $\leq \beta$ are precisely Bryan–Steinberg pairs for $\gamma > \gamma_s$. It follows that the generating series of Bryan–Steinberg invariants is obtained as a certain expansion of the rational function $f_\beta(q)$.

Step 3 and Step 4, i.e., the (γ, η)-wall-crossing, are illustrated in diagram 5.1.2.

Running through these four steps proves Theorem 5.1.1. The most concise statement of this theorem, encompassing all four steps, is now seen to be the following.

Theorem 5.1.5. Let $\mathcal{X}$ be a smooth CY3 orbifold satisfying the hard Lefschetz condition with projective coarse moduli space. For each multi-regular curve class $\beta \in \mathrm{N}_{1,\mathrm{mr}}(\mathcal{X})$ there exists a rational function $f_\beta(q)$ and an element $c_\beta \in \mathrm{N}_0(\mathcal{X})$ such that

1. the expansion of $f_\beta(q)$ with respect to $\mathrm{L}_{\mathrm{deg}}$ is the series $\mathrm{PT}(\mathcal{X})_\beta$,
2. the expansion of $f_\beta(q)$ with respect to some $\mathrm{L}_{\gamma'}$ is the series $\mathrm{PT}_f(\mathrm{Y}/\mathrm{X})_\beta$, and
3. for $\beta' \leq \beta$ and $\mathrm{A} \in \mathrm{Amp}(\mathrm{X})$ the locus $\{q^{2\beta' \cdot \mathrm{A}} - 1 = 0\}$ contains the poles of $f_\beta(q)$, where $\pi \colon \mathcal{X} \to \mathrm{X}$ denotes the coarse moduli space of $\mathcal{X}$.

Moreover, we conjecture the following corollary to the proof.

Conjecture 5.1.6. Let $\mathcal{X}$ be a smooth CY3 orbifold with projective coarse moduli space, and let $\beta \in \mathrm{N}_{\leq 1}(\mathcal{X})$ be a curve class. The rational function $f_\beta(q)$ satisfies

$$f_\beta(q) = f_{\beta^\vee}(q^\vee), \tag{5.1.10}$$

where $(-)^\vee \colon \mathbf{Q}[\mathrm{N}(\mathcal{X})] \to \mathbf{Q}[\mathrm{N}(\mathcal{X})]$ is the anti-isomorphism induced by $\mathbf{D}$. Note that if $\mathcal{X}$ satisfies the hard Lefschetz condition and β is multi-regular, there exists a class $c_\beta \in \mathrm{N}_0(\mathcal{X})$ such that $\beta^\vee = \beta + c_\beta$. In that case, the statement becomes

$$f_\beta(q) = q^{c_\beta} f_\beta(q^\vee). \tag{5.1.11}$$

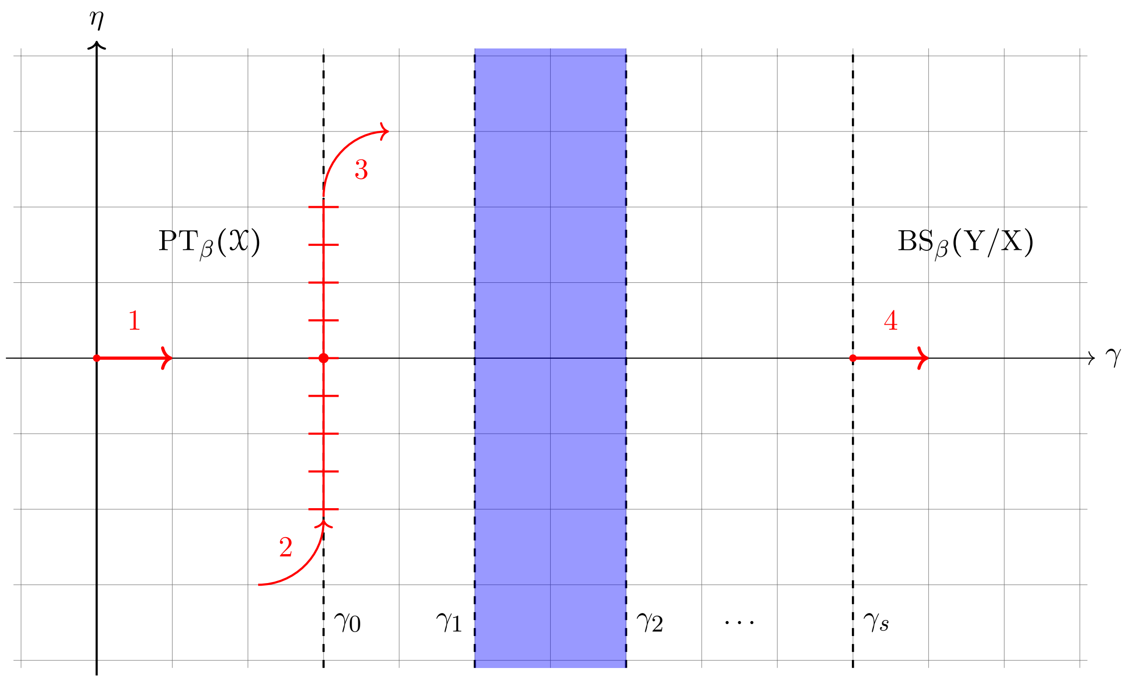


Figure 5.1: A schematic of the (γ, η)-wall-crossing. The notion of γ-pair is constant between two consecutive γ-walls, for example in the blue region of (γ, η) with $\gamma_1 < \gamma < \gamma_2$. Crossing all γ-walls $\gamma_1, \ldots, \gamma_s$ is done as follows: first, for $0 < \gamma \ll 1$ a γ-pair is a PT pair, regardless of the value of η. On the wall (γ_0, η), the notion of pair depends on η. The η-walls at γ_0 are indicated by horizontal red lines. Sending $\eta \to -\infty$ we slide off the wall to the left $(\gamma - \epsilon, \eta)$, whereas sending $\eta \to \infty$ we slide off the wall to the right $(\gamma + \epsilon, \eta)$ for some $\epsilon > 0$; see Proposition 5.3.31. Running the first wall-crossing in the other direction, we pass steps 2 and 3 and effectively cross the wall γ_0. Repeating this process for the walls $\gamma_1, \ldots, \gamma_s$, and identifying γ-pairs for $\gamma_s < \gamma$ with Bryan–Steinberg pairs, completes the argument.

5.2 Rationality of stable pair invariants

Let $\mathcal{X}$ be a smooth CY3 orbifold with projective coarse space X. We prove the rationality of the generating series of stable pair invariants $\mathrm{PT}(\mathcal{X})_\beta$ for any curve class $\beta \in \mathrm{N}_1(\mathcal{X})$. This result is motivated by a similar result for varieties as given in Conjecture 1.2.11.

We prove this result using the theory of wall-crossing and pairs as developed in the previous chapter. These notions will depend on (a shift of) the Nironi stability condition on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$. First, we establish suitable openness and finiteness properties of the moduli spaces involved. This allows us to apply the numerical wall-crossing formula. Second, we establish rationality of the series $\mathrm{PT}(\mathcal{X})_\beta$ using a periodicity of the counting invariants, induced by tensoring by an ample line bundle on X, and a duality of pairs induced by the derived dual. And third, we conjecture a symmetry of the rational re-summation of $\mathrm{PT}(\mathcal{X})_\beta$.

Remark 5.2.1. In this section alone, $\mathcal{X}$ denotes a general smooth CY3 orbifold with projective coarse moduli space $\pi: \mathcal{X} \to \mathrm{X}$. In particular, we do *not* assume that $\mathcal{X}$ satisfies the hard Lefschetz property.

We recall some notions from section 2.1.3 and establish some conventions.

1. We fix an ample line bundle A on X and a self-dual generating vector bundle V on $\mathcal{X}$, so $\mathrm{V} = \mathrm{V}^\vee$. Such a bundle exists by Lemma 2.1.30.
2. The modified Hilbert polynomial of a class $\mathrm{F} \in \mathrm{Coh}_{\le 1}(\mathcal{X})$ is
$$p_\mathrm{F}(k) := \chi(\mathrm{V}, \mathrm{F} \otimes \mathrm{A}^{\otimes k}) = a_1(\mathrm{F})k + a_0(\mathrm{F}) \in \mathbf{Z}[k]. \tag{5.2.1}$$
It only depends on the class of F in the numerical Grothendieck group $\mathrm{N}_{\le 1}(\mathcal{X})$.
3. The constant term of $p_\mathrm{F}(k)$ is the *degree of* F, i.e., $\deg(\mathrm{F}) := a_0(\mathrm{F}) = \chi(\mathrm{V}, \mathrm{F})$. It induces a linear function $\mathrm{L}_{\deg}: \mathrm{N}_0(\mathcal{X})_\mathbf{R} \to \mathbf{R}$ sending c to $\deg(c)$.
4. We fix an *integral* splitting $\mathrm{N}_{\le 1}(\mathcal{X}) = \mathrm{N}_1(\mathcal{X}) \oplus \mathrm{N}_0(\mathcal{X})$ *not necessarily* compatible with the degree, i.e., if a class splits as $\alpha = (\beta, c)$ it need *not* be the case that $\deg(\alpha) = \deg(c)$; we always mean $\deg(\beta, c)$. For a sheaf we write $[\mathrm{F}] = (\beta_\mathrm{F}, c_\mathrm{F})$.

5.2.1 Boundedness of Nironi stability

Recall the notion of Nironi stability from section 2.1.3. Given a sheaf $\mathrm{F} \in \mathrm{Coh}_{\le 1}(\mathcal{X})$, the *Nironi slope of* F is
$$\nu(\mathrm{F}) := \frac{a_0(\mathrm{F})}{a_1(\mathrm{F})} \in \mathbf{Q} \cup \{\infty\} \tag{5.2.2}$$
where $\nu(\mathrm{F}) = \infty$ only if $a_1(\mathrm{F}) = 0$. The latter occurs if and only if $\mathrm{F} \in \mathrm{Coh}_0(\mathcal{X})$. We refer to $\nu: \mathrm{N}_{\le 1}(\mathcal{X}) \to \mathbf{R} \cup \{\infty\}$ as the Nironi slope function. Given $\mathrm{F} \in \mathrm{Coh}_{\le 1}(\mathcal{X})$ we write $\mathrm{F}_{\max}$ for the semistable factor of F with the biggest slope in its Harder–Narasimhan filtration, which we denote by $\nu_{\max}(\mathrm{F})$. We employ analogous notation for $\mathrm{F}_{\min}$.

For any $\delta \in \mathbf{R}$, we can define a torsion pair on $\mathrm{Coh}_{\le 1}(\mathcal{X})$ by collapsing the Harder–Narasimhan filtration of ν into a two-term filtration split at the slope δ. We write

$$\begin{aligned} \mathtt{T}_\delta &:= \{\mathrm{T} \in \mathrm{Coh}_{\le 1}(\mathcal{X}) \mid \nu_{\min}(\mathrm{T}) \ge \delta\} = \{\mathrm{T} \in \mathrm{Coh}_{\le 1}(\mathcal{X}) \mid \mathrm{T} \twoheadrightarrow \mathrm{Q} \ne 0 \Rightarrow \nu(\mathrm{Q}) \ge \delta\} \\ \mathtt{F}_\delta &:= \{\mathrm{F} \in \mathrm{Coh}_{\le 1}(\mathcal{X}) \mid \nu_{\max}(\mathrm{F}) < \delta\} = \{\mathrm{F} \in \mathrm{Coh}_{\le 1}(\mathcal{X}) \mid 0 \ne \mathrm{S} \hookrightarrow \mathrm{F} \Rightarrow \nu(\mathrm{S}) < \delta\} \end{aligned}$$

for the induced family of numerical torsion pairs $(\mathtt{T}_\delta, \mathtt{F}_\delta)$ on $\mathrm{Coh}_{\le 1}(\mathcal{X})$.

Definition 5.2.2. We write $\mathtt{P}_\delta \subset \mathtt{A}$ for the category of $(\mathtt{T}_\delta, \mathtt{F}_\delta)$-pairs. For ease of notion, we also refer to $(\mathtt{T}_\delta, \mathtt{F}_\delta)$-pairs simply as *$\delta$-pairs*.

Remark 5.2.3. Varying δ affects the notion of $(\mathtt{T}_\delta, \mathtt{F}_\delta)$-pairs and, hence, the associated counting invariants. However, whether or not a sheaf in $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ is ν-(semi)stable is independent of δ.

We now prove two series of technical openness and finiteness results for the moduli spaces of objects defined via ν-stability. This will allow us to apply Theorem 4.3.16 to the associated counting invariants.

The first series of technical results concerns moduli of Nironi-semistable objects. Let $\mathrm{I} \subset \mathbf{R} \cup \{\infty\}$ denote an interval. Recall that $\mathcal{M}^{\mathrm{ss}}_\nu(\mathrm{I})$ denotes the full subcategory of sheaves $\mathrm{F} \in \mathrm{Coh}_{\leq 1}(\mathcal{X})$ such that $\nu_{\mathrm{min}}(\mathrm{F}), \nu_{\mathrm{max}}(\mathrm{F}) \in \mathrm{I}$, i.e., the slopes of all Nironi-semistable factors in the Harder–Narasimhan filtration of F lie in I. Finally, recall that the corresponding moduli stack is denote by $\underline{\mathcal{M}}^{\mathrm{ss}}_\nu(\mathrm{I})$; see section 2.1.4.

Proposition 5.2.4. Let $\delta \in \mathbf{R}$.

1. The torsion pair $(\mathtt{T}_\delta, \mathtt{F}_\delta)$ is open.
2. The stack $\underline{\mathcal{M}}^{\mathrm{ss}}_\nu(\delta)$ is open in $\underline{\mathrm{Coh}}_{\leq 1, \mathcal{X}}$, and $\mathcal{M}^{\mathrm{ss}}_\nu(\delta)$ is decompositionally finite.
3. For $\beta \in \mathrm{N}_1(\mathcal{X})$, consider the set

$$\mathcal{L}_\beta := \left\{ c \in \mathrm{N}_0(\mathcal{X}) \,\middle|\, \underline{\mathcal{M}}^{\mathrm{ss}}_\nu(\beta, c) \neq \varnothing \right\} \subset \mathrm{N}_0(\mathcal{X}). \tag{5.2.3}$$

   The image of $\mathcal{L}_\beta$ in $\mathrm{N}_0(\mathcal{X})/\mathbf{Z}(\beta \cdot \mathrm{A})$ is finite.

Proof. All three statements follow from Theorem 2.1.47. Indeed, for the first part we note that $\underline{\mathtt{T}}_\delta = \underline{\mathcal{M}}^{\mathrm{ss}}_\nu([\delta, \infty])$ and $\underline{\mathtt{F}}_\delta = \underline{\mathcal{M}}^{\mathrm{ss}}_\nu((-\infty, \delta))$. Similarly, for the second part, the substack $\underline{\mathcal{M}}^{\mathrm{ss}}_\nu(\delta, \beta) \subset \underline{\mathrm{Coh}}_{\mathcal{X}, \beta}$ is open and of finite type for any class $\beta \in \mathrm{N}_1(\mathcal{X})$. Hence $\mathcal{M}^{\mathrm{ss}}_\nu(\delta)$ defines an element of the graded Hall algebra $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$. Suppose we can decompose $(\beta, c) = (\beta', c') + (\beta'', c'')$ such that

$$\mathcal{M}^{\mathrm{ss}}_\nu(\beta', c') \neq \varnothing \neq \mathcal{M}^{\mathrm{ss}}_\nu(\beta'', c'') \tag{5.2.4}$$

and $\nu(\beta', c') = \delta = \nu(\beta'', c'')$. By Lemma 2.1.38, there are only finitely many effective classes $\beta', \beta'' \geq 0$ such that $\beta = \beta' + \beta''$. And given β', Theorem 2.1.47 shows that there are only finitely many choices for c' leading to non-empty moduli stacks. Furthermore, $\mathcal{M}^{\mathrm{ss}}_\nu(\delta)$ is closed under direct sums and summands by semistability. We conclude that $\mathcal{M}^{\mathrm{ss}}_\nu(\delta)$ is decompositionally finite.

Let $\beta \in \mathrm{N}_1(\mathcal{X})$ and write $d_\beta := a_1(\beta) \in \mathbf{Z}_{>0}$. For the third part, note that a sheaf $\mathrm{F} \in \mathrm{Coh}_{\leq 1}(\mathcal{X})$ such that $\beta_{\mathrm{F}} = \beta$ satisfies $d_\beta \nu(\mathrm{F}) \in \mathbf{Z}$. Consider the subset

$$\bigcup_{a=1}^{d_\beta} \left\{ c + \mathbf{Z}(\beta \cdot \mathrm{A}) \,\middle|\, \mathcal{M}^{\mathrm{ss}}_\nu(\beta, c) \neq \varnothing \text{ and } \nu(\beta, c) = \frac{a}{d_\beta} \right\} \subset \mathrm{N}_0(\mathcal{X})/\mathbf{Z}(\beta \cdot \mathrm{A}). \tag{5.2.5}$$

Each of the sets in this union is finite since the stack $\underline{\mathcal{M}}_\nu^{\mathrm{ss}}(a/d_\beta, \beta)$ is of finite type for every $a \in \mathbf{Z}$. This completes the proof. $\square$

Lemma 5.2.5. Let $\beta \in \mathrm{N}_1(\mathcal{X})$ and $\delta \in \mathbf{R}$. The set

$$\mathrm{S}_\delta := \{c \in \mathrm{N}_0(\mathcal{X}) \mid \mathcal{M}_\nu^{\mathrm{ss}}([\delta, \infty), (\beta, c)) \neq \varnothing\}. \tag{5.2.6}$$

is $\mathrm{L}_{\deg}$-bounded in the sense of Definition 2.5.7.

Proof. Take $\mathrm{M} \in \mathbf{R}$. We have to show that $\mathrm{S}_\delta \cap \{c \in \mathrm{N}_0(\mathcal{X}) \mid \mathrm{L}_{\deg}(c) \leq \mathrm{M}\}$ is finite. Note that $\mathrm{S}_{\delta'} \subseteq \mathrm{S}_\delta$ for all $\delta' \geq \delta$. Thus we may assume that $\delta < 0$.

Let $\mathrm{F} \in \mathcal{M}_\nu^{\mathrm{ss}}([\delta, \infty), (\beta, c))$, so $\beta_\mathrm{F} = \beta$ and $\nu_{\min}(\mathrm{F}) \geq \delta$. If $\nu_{\max}(\mathrm{F}) \leq 0$ is bounded for all such sheaves, it follows that

$$\mathrm{S}_\delta \subset \{c \in \mathrm{N}_0(\mathcal{X}) \bigm| \mathcal{M}_\nu^{\mathrm{ss}}([\delta, 0], (\beta, c)) \neq \varnothing\}. \tag{5.2.7}$$

But this is a finite set by Theorem 2.1.47. Thus we may assume that $\nu_{\max}(\mathrm{F}) > 0$.

Examining the ν-HN filtration of F, we find $\nu_{\max}(\mathrm{F}) \geq \nu(\mathrm{F}) \geq \nu_{\min}(\mathrm{F}) \geq \delta$. We establish an upper bound of $\nu_{\max}(\mathrm{F})$ in terms of $\deg(\beta, c)$, d_β, and δ. The exact sequence $\mathrm{F}_{\max} \hookrightarrow \mathrm{F} \twoheadrightarrow \mathrm{Q}$ yields $d_\beta = d_{\mathrm{F}_{\max}} + d_\mathrm{Q}$ all of which are positive integers because $\beta_{\mathrm{F}_{\max}}$ and β_Q are effective. Note that $\nu(\mathrm{Q}) \geq \nu_{\min}(\mathrm{F}) \geq \delta$. We have

$$\begin{aligned} \nu_{\max}(\mathrm{F}) \leq d_{\mathrm{F}_{\max}} \nu_{\max}(\mathrm{F}) &= d_\beta \nu(\mathrm{F}) - d_\mathrm{Q} \nu(\mathrm{Q}) \\ &\leq \deg(\mathrm{F}) - (d_\beta - d_{\mathrm{F}_{\max}})\delta \\ &\leq \deg(\beta, c) - (d_\beta - 1)\delta \end{aligned} \tag{5.2.8}$$

because $\delta < 0$ by assumption and $d_{\mathrm{F}_{\max}} \in \mathbf{Z}_{>0}$. We obtain the inclusion of sets

$$\mathrm{S}_\delta \cap \{c \in \mathrm{N}_0(\mathcal{X}) \bigm| \mathrm{L}_{\deg}(c) \leq \mathrm{M}\} \subset \big\{c \in \mathrm{N}_0(\mathcal{X}) \bigm| \mathcal{M}_\nu^{\mathrm{ss}}\big([\delta, \mathrm{M} - (d_\beta - 1)\delta], (\beta, c)\big) \neq \varnothing\big\}.$$

The latter is finite by Theorem 2.1.47, so we conclude that S_δ is $\mathrm{L}_{\deg}$-bounded. $\square$

The second series of technical results concerns moduli of δ-pairs, and how they change as δ varies. We collect a number of ingredients of these results.

Fix a class $\beta \in \mathrm{N}_1(\mathcal{X})$. First we locate the walls $w \in \mathbf{R}$ where the notion of δ-pair may change for objects of class $(-1, \beta', c') \in \mathrm{N}(\mathtt{A})$ with $\beta' \leq \beta$.

Definition 5.2.6. For $\beta \in \mathrm{N}_1(\mathcal{X})$, we set $\mathrm{W}_\beta = (1/d_\beta!)\mathbf{Z} \subset \mathbf{R}$ where $d_\beta = a_1(\beta) \in \mathbf{Z}_{>0}$.

Lemma 5.2.7. The notion of δ-pair of class $\beta' \leq \beta$ is locally constant for $\delta \in \mathbf{R} \setminus \mathrm{W}_\beta$.

Proof. Let E be a on object in $\mathtt{A}$ of class $(-1,\beta,c)$. By Corollary 4.1.9, we have $\mathrm{H}^{-1}(\mathrm{E}) = \mathrm{I}_{\mathrm{C}}$ for some curve $\mathrm{C} \subset \mathcal{X}$ and, writing $\mathrm{T} = \mathrm{H}^0(\mathrm{E})$, we have $\beta_{\mathrm{T}} \le \beta$. Moreover, we deduce that $d_\beta \ge d_{\beta_{\mathrm{T}}}$ because $\beta_{\mathrm{C}} = \beta - \beta_{\mathrm{T}}$ is effective. Thus $d_{\mathrm{C}} = \chi(\mathrm{V}, \mathrm{C}\cdot\mathrm{A}) \ge 0$ by Lemma 2.1.37 and the fact that $\mathrm{C}\cdot\mathrm{A} \in \mathrm{N}_0(\mathcal{X})$ is effective.

By definition, E is a δ-pair if and only if $\mathrm{Hom}(\mathtt{T}_\delta, \mathrm{E}) = 0 = \mathrm{Hom}(\mathrm{E}, \mathtt{F}_\delta)$. Let us discuss the second condition first. Note that $\mathrm{Hom}(\mathrm{E}, \mathtt{F}_\delta) = \mathrm{Hom}(\mathrm{T}, \mathtt{F}_\delta)$. Thus, the condition for E to be a pair changes when δ crosses the slope of T, which occurs in increments of $d_{\beta_{\mathrm{T}}}$. Since $d_{\beta_{\mathrm{T}}} \le d_\beta$, we will not see any change when $\delta \in \mathbf{R} \smallsetminus \mathrm{W}_\beta$ varies along an interval. The condition $\mathrm{Hom}(\mathtt{T}_\delta, \mathrm{E}) = 0$ follows similarly by dualising and noting that $\mathbf{D}(\mathrm{E})$ is a $(-\delta)$-pair of class $(-1,\beta^\vee,c^\vee)$ with $d_{\beta^\vee} = d_\beta$ by Lemma 5.2.9 below. $\square$

Let $p \in \mathbf{Z}[x]$ be a polynomial. Recall the fine moduli space $\mathrm{Quot}_{\mathcal{X}}(\mathrm{F}, p)$ representing the functor of quotients of $\mathrm{F} \in \mathrm{Coh}(\mathcal{X})$ of modified Hilbert polynomial p. This scheme is projective by Theorem 2.1.32. Furthermore, we write

$$\mathrm{Quot}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}, p) \coloneqq \mathrm{Quot}_{\mathcal{X}}(p) = \bigcup_{\substack{\alpha \in \mathrm{N}(\mathcal{X}) \\ p_\alpha = p}} \mathrm{Quot}_{\mathcal{X}}(\alpha) \tag{5.2.9}$$

where $\mathrm{Quot}_{\mathcal{X}}(\alpha)$ parametrises quotients of $\mathcal{O}_{\mathcal{X}}$ of numerical class α. Note that the projectivity implies that the set $\{\alpha \in \mathrm{N}(\mathcal{X}) \mid p_\alpha = p,\ \mathrm{Quot}_{\mathcal{X}}(\alpha) \ne \varnothing\}$ is finite.

The following result is key in showing boundedness for the moduli of pairs.

Lemma 5.2.8. Let $\beta \in \mathrm{N}_1(\mathcal{X})$ be a class. The set

$$\mathrm{Q}_{\le\beta} \coloneqq \bigcup_{\beta' \le \beta} \mathrm{Q}_{\beta'} \coloneqq \bigcup_{\beta' \le \beta} \{c \in \mathrm{N}_0(\mathcal{X}) \mid \mathrm{Quot}_{\mathcal{X}}(\beta', c) \ne \varnothing\} \tag{5.2.10}$$

is $\mathrm{L}_{\mathrm{deg}}$-bounded.

Proof. By Lemma 2.1.38, it suffices to prove that the set Q_β is $\mathrm{L}_{\mathrm{deg}}$-bounded for a fixed β. Recall that $p_{(\beta,c)}(k) = p_\beta(k) + \deg(c)$ for $c \in \mathrm{N}_0(\mathcal{X})$. For $d \in \mathbf{R}$, the projectivity of the Quot scheme implies that the subscheme

$$\bigcup_{c \in \mathrm{L}_{\mathrm{deg}}^{-1}(d)} \mathrm{Quot}_{\mathcal{X}}(\beta, c) \subseteq \mathrm{Quot}_{\mathcal{X}}(p_\beta + d) \tag{5.2.11}$$

is projective. It follows that the set $\mathrm{Q}_\beta \cap \mathrm{L}_{\mathrm{deg}}^{-1}(d)$ is finite.

Let $p = \mathrm{L}_{\mathrm{deg}}([\mathrm{pt}])$, where pt denotes a non-stacky point of $\mathcal{X}$. Since $\mathrm{L}_{\mathrm{deg}}(\mathrm{Q}_\beta) \subseteq \mathbf{Z}$, the scheme

$$\mathrm{H}_d = \bigcup_{c \in \mathrm{L}_{\mathrm{deg}}^{-1}([d, d+p])} \mathrm{Quot}_{\mathcal{X}}(\beta, c) \tag{5.2.12}$$

is projective, which means that $Q_\beta \cap L_{\deg}^{-1}([d, d+p])$ is finite. By adding on floating points, as explained in Remark 1.1.7, we see that $\dim H_{d-kp} \le \dim H_d - 3k$ for all $k \in \mathbf{Z}_{\ge 1}$. We conclude that $H_d = \varnothing$ for $d \ll 0$, and thus that Q_β is $L_{\deg}$-bounded. □

The following symmetry is another ingredient in the boundedness of δ-pairs. Recall the shifted derived dual $\mathbf{D}(-) = \mathbf{R}\underline{\mathrm{Hom}}(-, \mathcal{O}_\mathcal{X})[2]$. It acts as $\mathbf{D}(\mathrm{Coh}_1(\mathcal{X})) = \mathrm{Coh}_1(\mathcal{X})$ and $\mathbf{D}(\mathrm{Coh}_0(\mathcal{X})) = \mathrm{Coh}_0(\mathcal{X})[-1]$.

First, we record the following lemma.

Lemma 5.2.9. Let $F \in \mathrm{Coh}_{\le 1}(\mathcal{X})$ be one-dimensional and write $p_F(k) = a_1(F)k + a_0(F)$ for $k \in \mathbf{Z}$. We have $a_1(\mathbf{D}(F)) = a_1(F)$, $a_0(\mathbf{D}(F)) = -a_0(F)$, and $\nu(\mathbf{D}(F)) = -\nu(F)$.

Proof. Since $\mathbf{D}$ is an anti-equivalence and A is a line bundle, we find

$$p_{\mathbf{D}(F)}(k) = \chi(V, \mathbf{D}(F) \otimes A^{\otimes k}) = \chi(V, \mathbf{D}(F \otimes A^{\otimes -k})) = \chi(F \otimes A^{\otimes -k}, V^\vee). \quad (5.2.13)$$

It follows that $p_{\mathbf{D}(F)}(k) = -\chi(V, F \otimes A^{\otimes -k}) \equiv -p_F(-k)$ because V is a self-dual vector bundle and $\mathcal{X}$ is a Calabi–Yau threefold. The claimed properties follow directly. □

Lemma 5.2.10. Let $\delta \in \mathbf{R} \setminus \mathbf{Q}$. Let $E \in \mathtt{R}$. Then $E \in \mathtt{P}_\delta$ if and only if $\mathbf{D}(E) \in \mathtt{P}_{-\delta}$.

Proof. There is an asymmetry in the definition of pair: $T \in \mathtt{T}_\delta$ if $\nu(T) \ge \delta$ but $F \in \mathtt{F}_\delta$ if $\nu(F) < 0$. However, since slopes are rational, the symmetry is restored for irrational δ, i.e., $\nu(T) \ge \delta$ precisely if $\nu(T) > \delta$. It follows that $\mathbf{D}(\mathtt{F}_\delta) = \mathtt{T}_{-\delta} \cap \mathrm{Coh}_1(\mathcal{X})$ for $\delta \in \mathbf{R} \setminus \mathbf{Q}$.

Let $E \in P_\delta$ and note that $\mathrm{Coh}_0(\mathcal{X}) \subset \mathtt{T}_\delta$. Then $E \in \langle \mathcal{O}_\mathcal{X}[1], \mathrm{Coh}_0(\mathcal{X}), \mathrm{Coh}_1(\mathcal{X})\rangle_{\mathrm{ex}}$, and so it follows that $\mathbf{D}(E) \in \langle \mathcal{O}_\mathcal{X}[1], \mathrm{Coh}_1(\mathcal{X}), \mathrm{Coh}_0(\mathcal{X})[-1]\rangle_{\mathrm{ex}}$. Since $E \in \mathrm{Coh}_0(\mathcal{X})^\perp$ we have $\mathbf{D}(E) \in {}^\perp\mathrm{Coh}_0(\mathcal{X})[-1]$, and so $\mathbf{D}(E) \in \langle \mathcal{O}_\mathcal{X}[1], \mathrm{Coh}_1(\mathcal{X})\rangle_{\mathrm{ex}}$. Hence $\mathbf{D}(E) \in \mathtt{A}$.

Since $E \in (\mathtt{T}_\delta \cap \mathrm{Coh}_1(\mathcal{X}))^\perp$ we have $\mathbf{D}(E) \in {}^\perp\mathtt{F}_{-\delta}$, and since $E \in {}^\perp\mathtt{F}_\delta \cap {}^\perp\mathrm{Coh}_0(\mathcal{X})[-1]$ we have $E \in (\mathtt{T}_{-\delta} \cap \mathrm{Coh}_1(\mathcal{X}))^\perp \cap \mathrm{Coh}_0(\mathcal{X})^\perp = \mathtt{T}_{-\delta}^\perp$. We conclude that $\mathbf{D}(E) \in \mathbf{P}_{-\delta}$. □

We now come to the second set of technical results concerning moduli of pairs.

Proposition 5.2.11. Let $\delta \in \mathbf{R}$.

1. For any class $\beta \in N_1(\mathcal{X})$, the set $\{c \in N_0(\mathcal{X}) \mid \underline{P}_\delta(\beta, c) \ne \varnothing\}$ is finite.

2. For any class $(\beta, c) \in N_{\le 1}(\mathcal{X})$, the moduli stack $\underline{P}_\delta(\beta, c)$ is an open and finite type substack of $\mathfrak{Mum}_\mathcal{X}$.

3. There are only finitely many ways of decomposing a class $(\beta, c) \in N_{\le 1}(\mathcal{X})$ as $(\beta, c) = (\beta', c') + (\beta'', c'')$ with both $\mathtt{P}_\delta(\beta', c') \ne \varnothing$ and $\mathcal{M}_\nu^{\mathrm{ss}}(\beta'', c'') \ne \varnothing$.

Recall that $d_\beta = \chi(V, \beta \cdot A) \in \mathbf{Z}_{>0}$ for $0 \ne \beta \in N_1(\mathcal{X})$, and that $\nu(\beta, c) = \deg(c)/d_\beta$.

Proof. We may assume that $\delta \notin \mathbf{Q}$, otherwise replace δ with $\delta - \epsilon$ for $0 < \epsilon \ll 1$. Indeed, by the argument in the proof of Lemma 5.2.10, this does not change the notion of $(\mathtt{T}_\delta, \mathtt{F}_\delta)$-pair of curve class $\le \beta$. Furthermore, the derived dualising functor induces an automorphism of the stack $\mathfrak{Mum}_{\mathcal{X}}$ by Proposition 4.2.5. By the previous lemma, this restricts to an automorphism of stacks

$$\mathbf{D}: \underline{\mathrm{P}}_\delta \to \underline{\mathrm{P}}_{-\delta} \tag{5.2.14}$$

for $\delta \in \mathbf{R} \setminus \mathbf{Q}$. Thus we may additionally assume that $\delta > 0$.

For the first part, let E be a $(\mathtt{T}_\delta, \mathtt{F}_\delta)$-pair of class $(-1, \beta, c) \in \mathrm{N}(\mathtt{A}) = \mathbf{Z} \oplus \mathrm{N}_{\le 1}(\mathcal{X})$. Note that $\mathrm{H}^{-1}(\mathrm{E}) = \mathrm{I}_\mathrm{C}$ is the ideal sheaf of some at most one-dimensional substack $\mathrm{C} \subset \mathcal{X}$ by the argument in Lemma 5.2.7. Let $\mathrm{H}^0(\mathrm{E}) = \mathrm{T}$, so $\mathrm{T} \in \mathtt{T}_\delta$. We write $\deg(\mathrm{E}) = \chi(\mathrm{V}, \mathrm{E})$. It follows that

$$\deg(\mathrm{E}) = \deg(\mathcal{O}_\mathrm{C}) + \deg(\mathrm{T}). \tag{5.2.15}$$

If T is zero-dimensional, $\deg(\mathrm{T}) \ge 0$. If T is one-dimensional, $\deg(\mathrm{T}) \ge \nu(\mathrm{T}) \ge \delta$. By Lemma 5.2.8, there exists an integer N_β such that $\deg(\mathcal{O}_\mathrm{C}) \ge \mathrm{N}_\beta$ because $[\mathcal{O}_\mathrm{C}] \le \beta$. We deduce that $\deg(\mathrm{E}) \ge \mathrm{N}_\beta + \delta > \mathrm{N}_\beta$.

By Lemma 5.2.10, the object $\mathbf{D}(\mathrm{E})$ is a $(\mathtt{T}_{-\delta}, \mathtt{F}_{-\delta})$-pair, where $-\delta < 0$. Arguing similarly, we find $\deg(\mathbf{D}(\mathrm{E})) \ge -\delta d_{\beta^\vee} + \mathrm{M}_{\beta^\vee}$ for some integer $\mathrm{M}_{\beta^\vee}$ where $\beta^\vee \in \mathrm{N}_{\le 1}(\mathcal{X})$ is the dual class of β under $\mathbf{D}$. But $\deg(\mathrm{E}) = -\deg(\mathbf{D}(\mathrm{E}))$, so

$$\deg(\mathrm{E}) \le \delta d_{\beta^\vee} - \mathrm{M}_{\beta^\vee}, \tag{5.2.16}$$

from which we deduce $0 \le \deg(\mathrm{T}) \le \delta d_{\beta^\vee} - \mathrm{M}_{\beta^\vee} - \mathrm{N}_\beta$. By Lemma 5.2.5, it follows that there are only finitely many choices for $c_\mathrm{T} \in \mathrm{N}_0(\mathcal{X})$. Furthermore, combining the above inequalities with (5.2.15) also implies $\mathrm{N}_\beta \le \deg(\mathcal{O}_\mathrm{C}) \le \delta d_{\beta^\vee} - \mathrm{M}_{\beta^\vee}$. Hence there are only finitely many choices for $c_{\mathrm{I}_\mathrm{C}[1]} = c_{\mathcal{O}_\mathrm{C}}$ by Lemma 5.2.8. This proves part (1).

As for the finiteness in part (2), note that we have $0 \le \nu_{\max}(\mathrm{T}) \le \deg(\mathrm{T}) - (d_\beta - 1)\delta$. This follows by the argument of Lemma 5.2.5 and the fact that $\mathrm{T} \in \mathtt{T}_\delta$, i.e., $\nu(\mathrm{T}) \ge \delta$ is bounded below. Thus, by bounding $\deg(\mathrm{T})$ as we did just below equation (5.2.16), we find that $\nu_{\max}(\mathrm{T}) \le \mathrm{M}_\beta$ for some integer M_β.

Moreover, the exact sequences $\mathrm{I}_\mathrm{C}[1] \hookrightarrow \mathrm{E} \twoheadrightarrow \mathrm{T}$ and $\mathcal{O}_\mathrm{C} \hookrightarrow \mathrm{I}_\mathrm{C}[1] \twoheadrightarrow \mathcal{O}_{\mathcal{X}}[1]$ in $\mathtt{A}$ imply that $\mathcal{O}_\mathrm{C} \hookrightarrow \mathrm{E}$ is an injection in $\mathtt{A}$. Since E is a δ-pair, we have $\mathcal{O}_\mathrm{C} \in \mathtt{F}_\delta$ and so $\nu(\mathcal{O}_\mathrm{C}) < \delta$. In particular, $\deg(\mathcal{O}_\mathrm{C}) < d_{\mathcal{O}_\mathrm{C}}\delta \le d_\beta \delta$ because $[\mathcal{O}_\mathrm{C}] \le \beta$ and $\delta > 0$.

In conclusion, $\underline{\mathrm{P}}_\delta(\beta, c)$ is a substack of the stack of extensions of objects $\mathrm{I}_\mathrm{C}[1]$ in

$$\bigcup_{\beta' \le \beta} \bigcup_{\deg(c') \le d_\beta \delta} \mathrm{Quot}_{\mathcal{X}}(\beta', c') \tag{5.2.17}$$

by objects T in $\mathcal{M}^{\mathrm{ss}}_{\nu}([\delta, \mathrm{M}_\beta])$. The first stack is of finite type by Lemma 5.2.8 whereas the second is of finite type by Theorem 2.1.47. This proves that the stack $\underline{\mathtt{P}}_\delta(\beta, c)$ is of finite type.

Since $\mathrm{Coh}_0(\mathcal{X}) \subset \mathtt{T}_\delta$ for all $\delta \in \mathbf{R}$, we see that $\underline{\mathtt{P}}_\delta(\beta, c)$ is an open substack of $\mathfrak{Mum}_{\mathcal{X}}$ by Proposition 4.2.12 and part (1) of Proposition 5.2.4. This proves the second part.

As for the third part, the set $\{\beta' \in \mathrm{N}_1(\mathcal{X}) \,|\, \beta' \leq \beta\}$ is finite by Lemma 2.1.38. If $\mathtt{P}_\delta(\beta', c') \neq \varnothing$ then $\beta' \geq 0$ is effective by Corollary 4.1.9. We conclude by part (1). □

The following corollary encompasses the results of this section.

Corollary 5.2.12. Let $\beta \in \mathrm{N}_1(\mathcal{X})$, let $\delta \in \mathrm{W}_\beta$ be a wall for β, and let $0 < \epsilon \ll 1$ be such that $\mathrm{W}_\beta \cap [\delta - \epsilon, \delta + \epsilon] = \{\delta\}$. The torsion triple $(\mathtt{T}_{\delta+\epsilon}, \mathtt{W}_\delta, \mathtt{F}_{\delta-\epsilon})$ is open, numerical, and wall-crossing material in the sense of Definition 4.3.12.

Here $\mathtt{W}_\delta := \langle \mathrm{W} \in \mathrm{Coh}_{\leq 1}(\mathcal{X}) \mid \mathrm{W} \text{ is Nironi-semistable of } \nu(\mathrm{W}) = \delta \rangle_{\mathrm{ex}}$ denotes the subcategory of objects in $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ on the δ-wall.

Proof. The torsion triple is open by parts (1) and (2) of Proposition 5.2.4. It is numerical because it is induced by a stability function on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$. Note that $\mathrm{W}_\delta = \mathcal{M}^{\mathrm{ss}}_{\nu}(\delta)$. Thus the wall is decompositionally finite by part (2) of Proposition 5.2.4.

Part (2) of Proposition 5.2.11 precisely states that the subcategories $\mathtt{P}_{\delta\pm\epsilon}$ define elements in the graded Hall algebra $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$. To see that $\mathtt{P}_\pm = \mathtt{Pair}(\mathtt{T}_{\delta+\epsilon}, \mathtt{F}_{\delta-\epsilon})$ also defines an element, we reduce to the case $0 < \delta - \epsilon \in \mathbf{R} \smallsetminus \mathbf{Q}$ by applying the derived dual as in the previous Proposition. Such a hybrid pair fits in an exact sequence $\mathrm{I}_\mathrm{C}[1] \hookrightarrow \mathrm{E} \twoheadrightarrow \mathrm{T}$ in $\mathtt{A}$. We deduce the bounds $\nu(\mathrm{T}) \geq \delta + \epsilon$ and $\deg(\mathcal{O}_\mathrm{C}) \leq d_\beta(\delta - \epsilon)$ which is enough to conclude by the proof of part (2) of Proposition 5.2.11. Finally, part (3) of Proposition 5.2.11 now proves that $(\mathtt{T}_{\delta+\epsilon}, \mathtt{W}_\delta, \mathtt{F}_{\delta-\epsilon})$ is wall-crossing material. □

5.2.2 Counting invariants of δ-pairs

We are now in a position to apply the integration map to define DT-type invariants virtually counting Nironi-semistable sheaves and δ-pairs. We present these definitions for zero and non-zero rank separately.

Rank 0 Let $a \in \mathbf{R}$. By Lemma 5.2.4, the stack $\mathcal{M}^{\mathrm{ss}}_{\nu}(a)$ defines an element $\mathbf{1}_{\mathrm{SS}(a)}$ of the graded Hall algebra $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$, which is moreover decompositionally finite. In particular, we obtain a graded-regular element

$$\eta_{\mathrm{SS}(a)} := (\mathbf{L} - 1) \log \mathbf{1}_{\mathrm{SS}(a)} \in \mathrm{H}_{\mathrm{gr,reg}}(\mathtt{C}) \tag{5.2.18}$$

by Theorem 4.3.11. Projecting to the semi-classical quotient $\mathrm{H}_{\mathrm{gr,sc}}(\mathtt{C})$, we define DT-type invariants $\mathrm{J}_{(\beta,c)} \in \mathbf{Q}$ by the formula

$$\sum_{\nu(\beta,c)=a} \mathrm{J}_{(\beta,c)} z^{\beta} q^{c} := \mathrm{I}\left(\eta_{\mathrm{SS}(a)}\right) \in \mathbf{Q}\{\mathrm{N}(\mathcal{X})\}. \tag{5.2.19}$$

These invariants 'count' Nironi-semistable objects of slope a.

Rank -1 Let $(\beta, c) \in \mathrm{N}_{\leq 1}(\mathcal{X})$ be a class, and let $\delta \in \mathbf{R}$. By Corollary 4.2.21, we obtain an element $(\mathbf{L}-1)[\underline{\mathtt{P}}_{\delta}(\beta,c) \subset \underline{\mathtt{C}}] \in \mathrm{H}_{\mathrm{reg}}(\mathtt{C})$. Again, projecting to the semi-classical quotient and applying the integration morphism, we define DT-type invariants $\mathrm{DT}^{\delta}_{(\beta,c)} \in \mathbf{Z}$ via

$$\mathrm{DT}^{\delta}_{(\beta,c)} z^{\beta} q^{c} s := \mathrm{I}\big((\mathbf{L}-1)[\underline{\mathtt{P}}_{\delta}(\beta,c) \subset \underline{\mathtt{C}}]\big), \tag{5.2.20}$$

where $s = t^{-[\mathcal{O}_{\mathcal{X}}]}$ as before.

Remark 5.2.13. The crucial thing to notice is that the J-invariants do not depend on δ, whereas the invariants DT^{δ} *do* depend on δ. This is incorporated in their notation.

Fix a class $(\beta, c) \in \mathrm{N}_{\leq 1}(\mathcal{X})$. We now show that the invariant $\mathrm{DT}^{\delta}_{(\beta,c)}$ stabilises as δ tends to infinity, and that its limit equals the stable pair invariant $\mathrm{PT}_{\mathcal{X}}(\beta, c)$.

Lemma 5.2.14. For each $\alpha = (\beta, c) \in \mathrm{N}_{\leq 1}(\mathcal{X})$, there exists $\delta_\alpha \in \mathbf{R}$ such that $\mathrm{E} \in \mathtt{A}$ of class $(-1, \beta, c)$ is a $(\mathtt{T}_{\delta_\alpha}, \mathtt{F}_{\delta_\alpha})$-pair if and only if it is a $(\mathtt{T}_{\delta}, \mathtt{F}_{\delta})$-pair for all $\delta \geq \delta_\alpha$.

Proof. Let $0 < \epsilon \ll 1$. By the wall-crossing formula, the moduli stack $\underline{\mathtt{P}}_{\delta+\epsilon}(\alpha)$ can only differ from $\underline{\mathtt{P}}_{\delta}(\alpha) = \underline{\mathtt{P}}_{\delta-\epsilon}(\alpha)$ if the class α decomposes as $\alpha = \alpha' + \alpha''$ in $\mathrm{N}_{\leq 1}(\mathcal{X})$ such that the categories

$$\mathtt{P}_{\delta}(\alpha') \neq \varnothing \neq \mathcal{M}^{\mathrm{ss}}_{\nu}(\alpha'') \equiv \mathtt{W}(\delta) \tag{5.2.21}$$

where $\nu(\alpha'') = \delta$. Write these classes as $\alpha' = (\beta', c')$ and $\alpha'' = (\beta'', c'')$.

The set $\{\beta' \in \mathrm{N}_1(\mathcal{X}) \mid \beta' \leq \beta\}$ is finite by Lemma 2.1.38. Combined with part (1) of Proposition 5.2.11, there exists an integer N_β such that $\mathtt{P}_0(\beta', c') = \varnothing$ whenever (β', c') satisfies $\deg(c') \leq \mathrm{N}_\beta$ and $\beta' \leq \beta$. It follows that $\mathtt{P}_\delta(\beta', c') = \varnothing$ for all $\delta \geq 0$.

Suppose that there exists a decomposition as above consisting of a δ-pair E of class (β', c') and a Nironi semistable T of class (β'', c'') and slope $\nu(\mathrm{T}) = \delta$. Without loss of generality, we may take $\delta \geq 0$. This implies $\deg(c') > \mathrm{N}_\beta$. We find

$$\deg(c) = \deg(c') + \deg(c'') > \mathrm{N}_\beta + d_{\beta''}\nu(\mathrm{T}) \geq \mathrm{N}_\beta + \delta \tag{5.2.22}$$

because $d_{\beta''} \in \mathbf{Z}_{>0}$. Taking $\delta \geq \delta_\alpha := \max\{0, \deg(c) - \mathrm{N}_\beta\}$ completes the proof. □

Lemma 5.2.15. Let $\alpha = (\beta, c) \in \mathrm{N}_{\le 1}(\mathcal{X})$, and take $\mathrm{E} \in \mathtt{A}$ of class $(-1, \beta, c)$. If $\delta \ge \delta_{(\beta,c)}$, then E is a $(\mathtt{T}_\delta, \mathtt{F}_\delta)$-pair if and only if E is a PT pair.

Note that being a $(\mathtt{T}_\delta, \mathtt{F}_\delta)$-pair is independent of $\delta \ge \delta_{(\beta,c)}$ for E of class $(-1, \beta, c)$.

Proof. Assume that E is a PT pair, so $\mathrm{E} = (\mathcal{O}_{\mathcal{X}} \xrightarrow{s} \mathrm{F})$ with $\mathrm{coker}(s) \in \mathrm{Coh}_0(\mathcal{X})$ and $\mathrm{F} \in \mathrm{Coh}_1(\mathcal{X})$. If $\mathrm{S} \in \mathrm{Coh}_{\le 1}(\mathcal{X})$ is a subobject of E, then the inclusion factors through an inclusion $\mathrm{S} \hookrightarrow \mathrm{F}$ by Lemma 4.1.26. Hence $\nu_{\max}(\mathrm{S}) \le \nu_{\max}(\mathrm{F})$. Taking $\delta \ge \nu_{\max}(\mathrm{F})$, we find $\mathrm{S} \in \mathtt{F}_\delta$. Furthermore, if $\mathrm{Q} \in \mathrm{Coh}_{\le 1}(\mathcal{X})$ is a quotient object of E, it is a quotient of $\mathrm{coker}(s)$. Hence, $\mathrm{Q} \in \mathrm{Coh}_0(\mathcal{X}) \subset \mathtt{T}_\delta$ and E is a $(\mathtt{T}_\delta, \mathtt{F}_\delta)$-pair.

Conversely, suppose E is a $(\mathtt{T}_\delta, \mathtt{F}_\delta)$-pair. Set $\mathtt{T}_{\mathrm{PT}} = \mathrm{Coh}_0(\mathcal{X})$ and $\mathtt{F}_{\mathrm{PT}} = \mathrm{Coh}_1(\mathcal{X})$. By Lemma 4.1.16, it suffices to show that E is a $(\mathtt{T}_{\mathrm{PT}}, \mathtt{F}_{\mathrm{PT}})$-pair.

If $\mathrm{S} \in \mathrm{Coh}_{\le 1}(\mathcal{X})$ is a subobject of E, then S must be a pure one-dimensional sheaf. Hence $\mathrm{S} \in \mathtt{F}_{\mathrm{PT}}$. Furthermore, let G be the pure one-dimensional part of $\mathrm{H}^0(\mathrm{E})$, which is a quotient object of E in $\mathtt{A}$. If $\mathrm{G} \ne 0$, then taking $\delta > \nu(\mathrm{G})$ implies $\mathrm{G} \notin \mathtt{T}_\delta$, which contradicts E being a $(\mathtt{T}_\delta, \mathtt{F}_\delta)$-pair. Hence $\mathrm{G} = 0$, and so $\mathrm{H}^0(\mathrm{E}) \in \mathrm{Coh}_0(\mathcal{X}) = \mathtt{T}_{\mathrm{PT}}$. □

The following consequence of the above results is now immediate.

Corollary 5.2.16. Let $(\beta, c) \in \mathrm{N}_{\le 1}(\mathcal{X})$. If $\delta \ge \delta_{(\beta,c)}$, then $\mathrm{DT}^\delta_{(\beta,c)} = \mathrm{PT}_{\mathcal{X}}(\beta, c)$.

This corollary establishes the link between δ-pair counts and stable pair counts.

5.2.3 The proof of rationality

With all the boundedness results in place, we now apply the numerical wall-crossing formula to prove the rationality of the generating series of stable pair invariants.

First, we describe the ring in which this statement holds. To understand why we work in this ring, we make some elementary remarks about wall-crossing.

1. The set of all walls $\bigcup_{\beta \in \mathrm{N}_1(\mathcal{X})} \mathrm{W}_\beta \subset \mathbf{R}$ where the notion of δ-pair may change, is dense. To be able to apply the wall-crossing formula, we should fix a curve class.

2. Fix a class $\alpha = (\beta, c) \in \mathrm{N}_{\le 1}(\mathcal{X})$, let $\delta \in \mathrm{W}_\beta$ be a wall, and let $0 < \epsilon \ll 1$ such that $[\delta - \epsilon, \delta + \epsilon] \cap \mathrm{W}_\beta = \{\delta\}$. The key remark is that crossing the wall δ can only produce a $(\delta + \epsilon)$-pair of class $(-1, \beta, c)$ if it arises as an extension of a $(\delta - \epsilon)$-pair of class $(-1, \beta', c')$ and a Nironi semistable sheaf of slope δ and class (β'', c'').

   Since $\beta', \beta'' \le \beta$, it follows that curve classes γ such that $\gamma \nleq \beta$ do not come into play when considering (the generating function of) δ-pairs of class $(-1, \beta, c)$.

In conclusion, we are interested in the following elements in $\mathbf{Q}\{\mathrm{N}(\mathcal{X})\}$:

$$\mathrm{DT}^{\delta}_{\le\beta} := \sum_{\beta'\le\beta}\sum_{c\in\mathrm{N}_0(\mathcal{X})} \mathrm{DT}^{\delta}_{(\beta',c)} z^{\beta'} q^{c}, \tag{5.2.23}$$

$$\mathrm{J}(\delta)_{\le\beta} := \sum_{\beta'\le\beta}\sum_{\substack{c\in\mathrm{N}_0(\mathcal{X})\\ \nu(\beta',c)=\delta}} \mathrm{J}_{(\beta',c)} z^{\beta'} q^{c}. \tag{5.2.24}$$

Note that $\mathrm{DT}^{\delta}_{(\beta',c')}$ can only be non-zero for an effective class $0 \le \beta' \le \beta$ by Corollary 4.1.9. There are finitely many of such by Lemma 2.1.38. By part 1 of Proposition 5.2.11, we deduce $\mathrm{DT}^{\delta}_{\le\beta} \in \mathbf{Q}[\mathrm{N}(\mathcal{X})]$. Similarly, $\mathrm{J}(\delta)_{\le\beta} \in \mathbf{Q}[\mathrm{N}(\mathcal{X})]$.

However, we can be more precise. Recall $\Lambda \subset \mathrm{N}_1(\mathcal{X})$ the *effective cone of curves*, the commutative monoid spanned by classes of one-dimensional sheaves. Let $\Gamma \subset \mathrm{N}(\mathtt{A}) = \mathrm{N}_{\le1}(\mathcal{X}) \oplus \mathbf{Z}$ denote the *effective cone* generated by numerical classes of objects in $\mathtt{A}$.

Definition 5.2.17. Define $\mathbf{Q}[\Gamma] \subset \mathbf{Q}[\mathrm{N}(\mathcal{X})]$ as the vector space with $\mathbf{Q}$-basis

$$\mathrm{S} = \left\{ z^{\beta} q^{c} s^{k} \in \mathbf{Q}[\mathrm{N}(\mathcal{X})] \,\middle|\, \beta \in \Lambda,\ c \in \mathrm{N}_0(\mathcal{X}),\ k \in \mathbf{Z}_{\ge0} \right\} \tag{5.2.25}$$

where $s = t^{-[\mathcal{O}_{\mathcal{X}}]}$. Note that $\mathbf{Q}[\Gamma]$ define a Poisson subalgebra of $\mathbf{Q}[\mathrm{N}(\mathcal{X})]$.

Let $\beta \in \mathrm{N}_1(\mathcal{X})$ be a class. Consider the ideal $\mathrm{I}_{\beta} \subset \mathbf{Q}[\Gamma]$ generated by $\{z^{\beta'}, s^2 \mid \beta' \nleq \beta\}$. By the above discussion, we lose nothing by working in the quotient $\mathbf{Q}[\Gamma]_{\beta} := \mathbf{Q}[\Gamma]/\mathrm{I}_{\beta}$. Indeed, we can recover all coefficients of a power series in $\mathbf{Q}[\Gamma]$ supported outside of I_{β} via its image in $\mathbf{Q}[\Gamma]_{\beta}$. Note that $\mathbf{Q}[\Gamma] \twoheadrightarrow \mathbf{Q}[\Gamma]_{\beta}$ is a Poisson algebra morphism.

Proposition 5.2.18. Let $\beta \in \mathrm{N}_1(\mathcal{X})$ be a class, and let $\delta \in \mathrm{W}_{\beta}$. There exists an $\epsilon > 0$ such that $[\delta-\epsilon, \delta+\epsilon] \cap \mathrm{W}_{\beta} = \{\delta\}$. Moreover, the identity

$$\mathrm{DT}^{\delta+\epsilon}_{\le\beta} s = \exp(\{\mathrm{J}(\delta)_{\le\beta}, -\})\mathrm{DT}^{\delta-\epsilon}_{\le\beta} s \tag{5.2.26}$$

holds in the Poisson algebra $\mathbf{Q}[\Gamma]_{\beta}$, the quotient of $\mathbf{Q}[\Gamma] \subset \mathbf{Q}\{\mathrm{N}(\mathcal{X})\}$.

Proof. Since the walls for δ-pairs form a dense subset of $\mathbf{R}$, we proceed with some care. Define the full subcategory $\mathtt{T}'_{\delta} := \{\mathrm{T} \in \mathrm{Coh}_{\le1}(\mathcal{X}) \mid \nu_{\min}(\mathrm{T}) > \delta\}$; the only difference with $\mathtt{T}_{\delta}$ is the strict inequality. Similarly, define its perpendicular subcategory $\mathtt{F}'_{\delta}$ such that $(\mathtt{T}'_{\delta}, \mathtt{F}'_{\delta})$ is a torsion pair on $\mathrm{Coh}_{\le1}(\mathcal{X})$. We write $\mathtt{P}'_{\delta} := \mathtt{Pair}(\mathtt{T}'_{\delta}, \mathtt{F}'_{\delta})$.

The torsion triple $(\mathtt{T}'_{\delta}, \mathtt{W}_{\delta}, \mathtt{F}_{\delta})$ is open, numerical, and wall-crossing material by the results of Corollary 5.2.12. Here $\mathtt{W}_{\delta} = \mathcal{M}^{\mathrm{ss}}_{\nu}(\delta)$ is the category of Nironi-semistable sheaves of slope δ. Thus, we may apply Theorem 4.3.16 to obtain the identity

$$\mathrm{I}_{\mathrm{gr}}\Big((\mathbf{L}-1)[\underline{\mathtt{P}}'_{\delta} \subset \underline{\mathtt{C}}]\Big) = \exp(\{\mathrm{J}(\delta), -\})\mathrm{I}_{\mathrm{gr}}\Big((\mathbf{L}-1)[\underline{\mathtt{P}}_{\delta} \subset \underline{\mathtt{C}}]\Big) \tag{5.2.27}$$

in $\mathbf{Q}\{\Gamma\} \subset \mathbf{Q}\{\mathrm{N}(\mathcal{X})\}$. Here $\mathrm{J}(\delta) := \sum \mathrm{J}_{(\beta,c)} z^\beta q^c$ where the sum runs over $(\beta, c) \in \mathrm{N}_{\leq 1}(\mathcal{X})$ of slope $\nu(\beta, c) = \delta$. Projecting this identity to the quotient $\mathbf{Q}\{\Gamma\}_\beta$ sets those coefficients with $\beta' \nleq \beta$ to zero. Hence there exists an $0 < \epsilon \ll 1$ as claimed. But then $\mathtt{T}'_\delta = \mathtt{T}_{\delta+\epsilon}$, $\mathtt{F}_\delta = \mathtt{F}_{\delta-\epsilon}$, and $\mathtt{P}'_\delta = \mathtt{P}_{\delta+\epsilon}$. Thus, the formula reduces to equation (5.2.26) in $\mathbf{Q}[\Gamma]_\beta$. □

We now prove the rationality of the generating series of stable pair invariants.

Theorem 5.2.19. For each class $\beta \in \mathrm{N}_1(\mathcal{X})$, multi-regular or not, there exists a rational function $f_\beta(q)$ such that the series

$$\mathrm{PT}(\mathcal{X})_\beta = \sum_{c \in \mathrm{N}_0(\mathcal{X})} \mathrm{PT}_{\mathcal{X}}(\beta, c) q^c \tag{5.2.28}$$

is the expansion in $\mathbf{Q}[\mathrm{N}_0(\mathcal{X})]_{\mathrm{L_{deg}}}$ of $f_\beta(q)$.

More precisely, we can write $f_\beta(q)$ as a sum of functions $f_{\mathrm{D}}/g_{\mathrm{D}}$, where D is a decomposition $\beta = \sum_{i=1}^r \beta_i$ into effective classes, where $f_{\mathrm{D}} \in \mathbf{Z}[\mathrm{N}_0(\mathcal{X})]$, and where

$$g_{\mathrm{D}} = \prod_{i=1}^r (1 - \prod_{j=1}^i q^{2\beta_j \cdot \mathrm{A}})^{2i}. \tag{5.2.29}$$

The proof below is quite notation-heavy. To clarify the deluge of indices which follow, it is instructive to perform a toy computation. The root of the notational mess is the operator $\exp(\{\mathrm{J}(\delta)_{\leq\beta}, -\})$. Let $0 < \delta \in \mathrm{W}_\beta$ be the smallest wall. Expanding the Poisson bracket, we have

$$\begin{aligned}
\left\{\mathrm{J}(\delta)_{\leq\beta}, \mathrm{DT}^0_{\leq\beta} s\right\} &= \left\{ \sum_{\substack{\beta' \leq \beta \\ c' \in \mathrm{N}_0(\mathcal{X}) \\ \nu(\beta',c')=\delta}} \mathrm{J}_{(\beta',c')} z^{\beta'} q^{c'}, \sum_{\substack{\beta'' \leq \beta \\ c'' \in \mathrm{N}_0(\mathcal{X})}} \mathrm{DT}^0_{(\beta'',c'')} z^{\beta''} q^{c''} s \right\} \\
&= \sum_{\substack{\beta',\beta'' \leq \beta \\ c',c'' \in \mathrm{N}_0(\mathcal{X}) \\ \nu(\beta',c')=\delta}} \mathrm{J}_{(\beta',c')} \mathrm{DT}^0_{(\beta'',c'')} \left\{ z^{\beta'} q^{c'}, z^{\beta''} q^{c''} s \right\}
\end{aligned}$$

and

$$\left\{ z^{\beta'} q^{c'}, z^{\beta''} q^{c''} s \right\} = \pm\chi((\beta', c'), (\beta'', c'') - [\mathcal{O}_{\mathcal{X}}]) z^{\beta'+\beta''} q^{c'+c''} s,$$

where the sign is determined by $\sigma^{\chi((\beta',c'),(\beta'',c'')-[\mathcal{O}_{\mathcal{X}}])}$. Recall that the choice of sign $\sigma = +1$ corresponds to topological Euler characteristics, whereas the choice $\sigma = -1$ yields their Behrend weighted analogue. Thus, for stable pair invariants, the reader should read $\sigma = -1$.

Going one term further in the exponential, we see

$$\left\{\mathrm{J}(\delta)_{\leq\beta}, \left\{\mathrm{J}(\delta)_{\leq\beta}, \mathrm{DT}^0_{\leq\beta} s\right\}\right\} = \sum_{\substack{\beta_1,\beta_2,\beta'\leq\beta \\ c_1,c_2,c'\in\mathrm{N}_0(\mathfrak{X}) \\ \nu(\beta_i,c_i)=\delta}} \mathrm{J}_{(\beta_1,c_1)}\mathrm{J}_{(\beta_2,c_2)}\mathrm{DT}^0_{(\beta',c')}\,\mathrm{B}\, z^{\beta_1+\beta_2+\beta'} q^{c_1+c_2+c'} s$$

where each coefficient B, depending on $\beta_1, \beta_2, \beta' \leq \beta$ and $c_1, c_2, c' \in \mathrm{N}_0(\mathfrak{X})$, is given by

$$\mathrm{B} = \pm\chi((\beta_1, c_1), (\beta_2 + \beta', c_2 + c') - [\mathcal{O}_{\mathfrak{X}}])\chi((\beta_2, c_2), (\beta', c') - [\mathcal{O}_{\mathfrak{X}}])$$

and the sign is given by

$$\sigma^{\chi((\beta_1,c_1),(\beta_2+\beta',c_2+c')-[\mathcal{O}_{\mathfrak{X}}])+\chi((\beta_2,c_2),(\beta',c')-[\mathcal{O}_{\mathfrak{X}}])}.$$

Finally, we furthermore need to iterate the wall-crossing formula, which means composing the operators $\exp(\{\mathrm{J}(\delta)_{\leq\beta}, -\})$ as δ increases towards infinity.

Proof of Theorem 5.2.19. Iterating the wall-crossing formula from Proposition 5.2.18,

$$\mathrm{DT}^\infty_{\leq\beta} s = \prod_{\delta\in\mathrm{W}_\beta\cap[0,\infty)} \exp(\{\mathrm{J}(\delta)_{\leq\beta}, -\})(\mathrm{DT}^0_{\leq\beta} s),$$

where the product is taken in increasing order of δ. Substituting Definition (5.2.24) of $\mathrm{J}(\delta)_{\leq\beta}$, and expanding the exponential, the $z^\beta s$-coefficient of the right hand side becomes an infinite sum. The terms of this sum are described as follows. Fix an $r \geq 0$, a sequence $(\alpha_i)_{i=1}^r = (\beta_i, c_i)_{i=1}^r \subset \mathrm{N}_{\leq 1}(\mathfrak{X})$, and a class $\alpha' = (\beta', c') \in \mathrm{N}_{\leq 1}(\mathfrak{X})$, satisfying

- $\beta = \beta' + \sum \beta_i$,
- $\delta \leq \nu(\alpha_1) \leq \nu(\alpha_2) \leq \cdots \leq \nu(\alpha_r)$,
- $\mathrm{J}_{\alpha_i} \neq 0$ for all $1 \leq i \leq r$,
- $\mathrm{DT}^\delta_{\alpha'} \neq 0$.

The non-zero term in the infinite sum associated with this data is

$$\mathrm{T}((\alpha_i), \alpha') z^\beta q^{c'+\sum c_i} s = \mathrm{A}_{(\alpha_i)}\{\mathrm{J}_{\alpha_r} z^{\beta_r} q^{c_r}, -\} \circ \cdots \circ \{\mathrm{J}_{\alpha_1} z^{\beta_1} q^{c_1}, -\}(\mathrm{DT}^0_{\alpha'} z^{\beta'} q^{c'} s),$$

where $\mathrm{A}_{(\alpha_i)}$ is a factor arising from the exponential:

$$\mathrm{A}_{(\alpha_i)} := \prod_{\delta \in \mathrm{W}_\beta} \frac{1}{|\{i \mid \nu(\alpha_i) = \delta\}|!}. \tag{5.2.30}$$

Putting all these terms together, we conclude that

$$\mathrm{DT}_\beta^\infty = \sum_{r,(\alpha_i),\alpha'} \mathrm{T}(r,(\alpha_i),\alpha') z^\beta q^{c'+\sum c_i}. \tag{5.2.31}$$

This sum equals the generating series $\mathrm{PT}(\mathcal{X})_\beta$ by Corollary 5.2.16. We claim that it is the expansion of a rational function with respect to $\mathrm{L}_{\mathrm{deg}}$.

To see this, we write out the formula for the Poisson bracket. This yields

$$\mathrm{T}(r,(\alpha_i),\alpha') = \mathrm{A}_{(\alpha_i)} \mathrm{B}_{(\alpha_i),\alpha'} \left(\prod_{i=1}^r \mathrm{J}_{\alpha_i} \right) \mathrm{DT}^0_{\alpha'} \tag{5.2.32}$$

where

$$\mathrm{B}_{(\alpha_i),\alpha'} = \sigma^{\sum_{i<j} \chi(\alpha_j,\alpha_i) + \sum_i \chi(\alpha_i,\alpha'-[\mathcal{O}_\mathcal{X}])} \prod_{i=1}^r \chi(\alpha_i, -[\mathcal{O}_\mathcal{X}] + \alpha' + \sum_{j=1}^{i-1} \alpha_j). \tag{5.2.33}$$

We emphasize that the precise formula for $\mathrm{B}_{(\alpha_i),\alpha'}$ is less important. The important point is that it depends quasi-polynomially on the classes α_i. Indeed, together with the invariance of the J_{α_i} under tensoring by the line bundle A, the quasi-polynomiality of B yields periodic behaviour of the $\mathrm{T}(r,(\alpha_i),\alpha')$ as we now show.

We partition these T-terms in groups as follows. A *group* consists of the data of a class $\alpha' = (\beta', c')$, a sequence $(\beta_i)_{i=1}^r$, a sequence $\{\kappa_i\}_{i=1}^r$ where $\kappa_i \in \mathrm{N}_0(\mathcal{X})/\mathbf{Z}(\beta_i \cdot \mathrm{A})$, and a subset $\mathrm{E} \subseteq \{1,\dots,r-1\}$. This data is required to satisfy the conditions

- $\beta = \beta' + \sum_{i=1}^r \beta_i$, and
- $\mathrm{J}_{(\beta_i,c_i)} \neq 0$ for $c_i \in \kappa_i$ and all $i = 1,2,\dots,r$.

Note that for any class $(\gamma, d) \in \mathrm{N}_1(\mathcal{X}) \oplus \mathrm{N}_0(\mathcal{X})$, tensoring by the line bundle A induces an isomorphism $\underline{\mathcal{M}}_\nu^{\mathrm{ss}}(\gamma, d) \cong \underline{\mathcal{M}}_\nu^{\mathrm{ss}}(\gamma, d + \gamma \cdot \mathrm{A})$. As a consequence, the invariant $\mathrm{J}_{(\beta_i,c_i)}$ is independent of the choice of representative $c_i \in \kappa_i$. Thus we may write $\mathrm{J}_{(\beta_i,\kappa_i)} := \mathrm{J}_{(\beta_i,c_i)}$.

Collecting all terms belonging to the group $(\alpha', (\beta_i), (\kappa_i), \mathrm{E})$, we obtain

$$\mathrm{C}(\alpha', (\beta_i), (\kappa_i), \mathrm{E}) = \sum_{c_i} \mathrm{T}(r, ((\beta_i, c_i)), \alpha') q^{c'+\sum c_i}, \tag{5.2.34}$$

where the sum is over all $c_i \in \mathrm{N}_0(\mathcal{X})$ such that

$$c_i \in \kappa_i, \tag{5.2.35}$$
$$0 < \nu(\beta_1, c_1) \leq \nu(\beta_2, c_2) \cdots \leq \nu(\beta_r, c_r), \tag{5.2.36}$$
$$\nu(\beta_i, c_i) = \nu(\beta_{i+1}, c_{i+1}) \Leftrightarrow i \in \mathrm{E}. \tag{5.2.37}$$

Note that for such a choice of c_i, the factor $\mathrm{A}_{((\beta_i, c_i))}$ defined above depends only on E. Indeed, set $\{n_i\} = \{1, \dots, r\} \smallsetminus \mathrm{E}$ with $n_1 < n_2 < \dots < n_{r-|\mathrm{E}|}$. Then

$$\mathrm{A}_\mathrm{E} := \prod \frac{1}{(n_i - n_{i-1})!} = \mathrm{A}_{((\beta_i, c_i))}. \tag{5.2.38}$$

We find that the contribution of the group $(\alpha', (\beta_i), (\kappa_i), \mathrm{E})$ is

$$\mathrm{C}(\alpha', (\beta_i), (\kappa_i), \mathrm{E}) = \mathrm{A}_\mathrm{E} \prod_{i=1}^{r} \mathrm{J}_{\beta_i, \kappa_i} \mathrm{DT}^0_{\alpha'} \left(\sum_{c_i} \mathrm{B}_{(\beta_i, c_i), \alpha'} q^{c' + \sum c_i} \right) \tag{5.2.39}$$

where the sum runs over all $c_i \in \mathrm{N}_0(\mathcal{X})$ as above. Now, for every choice of (β_i), (κ_i), and E, there exists a sequence (c_i^0) with $c_i^0 \in \kappa_i$ which is *minimal* in the sense that replacing any c_i^0 with $c_i^0 - \beta_i \cdot \mathrm{A}$ would violate one of (5.2.36) and (5.2.37). We find

$$\mathrm{C}(\alpha', (\beta_i), (\kappa_i), \mathrm{E}) = \mathrm{A}_\mathrm{E} \prod_{i=1}^{r} \mathrm{J}_{\beta_i, \kappa_i} \mathrm{DT}^0_{\alpha'} \left(\sum_{a_i} \mathrm{B}_{(\beta_i, c_i^0 + a_i \beta_i \cdot \mathrm{A}), \alpha'} q^{c' + \sum c_i^0 + a_i \beta_i \cdot \mathrm{A}} \right) \tag{5.2.40}$$

where the sum is over the set $\mathrm{S}_\mathrm{E} = \{0 \leq a_1 \leq a_2 \leq \dots \leq a_r \mid a_i \in \mathbf{Z},\ a_i = a_{i+1} \Leftrightarrow i \in \mathrm{E}\}$.

Since the Euler form is bilinear, we conclude by equation (5.2.33) that B depends quasi-polynomially on the a_i with quasi-period 2 (because of σ). Lemma 2.5.15 shows

$$\mathrm{C}(\alpha', (\beta_i), (\kappa_i), \mathrm{E}) = \frac{p}{\prod_{i \in [r] \smallsetminus \mathrm{E}} (1 - \prod_{j=1}^{i} q^{2\beta_j \cdot \mathrm{A}})^{2i}} \tag{5.2.41}$$

holds in $\mathbf{Q}[\mathrm{N}_0(\mathcal{X})]_{\mathrm{L}_{\deg}}$ for some Laurent polynomial $p \in \mathbf{Q}[\mathrm{N}_0(\mathcal{X})]$, the exponent is $2i$ because $\sum_{j \geq r-i+1} \deg_{a_i} \mathrm{B} \leq 2i - 1$, and where we write $[r] = \{1, 2, \dots, r\}$.

Finally, we claim that there are only finitely many such *groups*, i.e., there are only finitely many non-trivial choices for the data of $(\alpha', (\beta_i), (\kappa_i), \mathrm{E})$. The sum of those rational functions is then $f_\beta(q)$. For the choice of (β_i) and E, this is obvious. The claim for α' follows from part 1 of Proposition 5.2.11, and the claim for (κ_i) follows from part 3 of Proposition 5.2.4.

In conclusion, the series DT^∞_β is the sum of DT^0_β, which is a Laurent polynomial by part 1 of Proposition 5.2.11, and finitely many terms as in equation (5.2.41). □

We deduce constraints on the location of the poles of the rational function $f_\beta(q)$.

Corollary 5.2.20. Let $\mathcal{X}$ be a CY3 orbifold with projective coarse moduli space X, let $\beta \in \mathrm{N}_1(\mathcal{X})$. The poles of the rational function $f_\beta(q)$ lie on the locus $\{q^{2\beta'\cdot \mathrm{A}} - 1 = 0\}$ where $\beta' \le \beta$ and $\mathrm{A} \in \mathrm{Amp}(\mathrm{X})$, and potentially at $q = 0$.

Proof. This follows from (5.2.41) and the fact that DT^0_β is a Laurent polynomial. □

We conjecture that the rational function $f_\beta(q)$ has a symmetry. To be precise, we conjecture the following.

Conjecture 5.2.21. Let $\mathcal{X}$ be a smooth CY3 orbifold with projective coarse moduli space, and let $\beta \in \mathrm{N}_{\le 1}(\mathcal{X})$ be a curve class. The rational function $f_\beta(q)$ satisfies

$$f_\beta(q) = f_{\beta^\vee}(q^\vee), \tag{5.2.42}$$

where $(-)^\vee\colon \mathbf{Q}[\mathrm{N}(\mathcal{X})] \to \mathbf{Q}[\mathrm{N}(\mathcal{X})]$ is the anti-isomorphism induced by $\mathbf{D}$. Note that if $\mathcal{X}$ satisfies the hard Lefschetz condition and β is multi-regular, there exists a class $c_\beta \in \mathrm{N}_0(\mathcal{X})$ such that $\beta^\vee = \beta + c_\beta$. In that case, the statement becomes

$$f_\beta(q) = q^{c_\beta} f_\beta(q^\vee), \tag{5.2.43}$$

They key ingredients are the following.

1. For each class $\beta \in \mathrm{N}_1(\mathcal{X})$, the series

$$\mathrm{DT}^{-\infty}_\beta = \sum_{c \in \mathrm{N}_0(\mathcal{X})} \mathrm{DT}^{-\infty}_{(\beta,c)} q^c \tag{5.2.44}$$

   is the expansion in $\mathbf{Q}[\mathrm{N}_0(\mathcal{X})]_{-\mathrm{L}_{\deg}}$ of the rational function $f_\beta(q)$.

2. Equality (5.2.14) stating $\mathbf{D}\colon \underline{\mathrm{P}}_\delta(\alpha) \cong \underline{\mathrm{P}}_{-\delta}(\alpha^\vee)$ as stacks for $\delta \in \mathbf{R} \smallsetminus \mathbf{Q}$.

3. Given a class $\alpha \in \mathrm{N}_{\le 1}(\mathcal{X})$, there exists δ_α such that $\delta \ge \delta_\alpha$ implies $\mathrm{DT}^\delta_\alpha = \mathrm{PT}_\mathcal{X}(\alpha)$. So by self-duality, if $-\delta \le -\delta_{\alpha^\vee}$ then $\mathrm{DT}^{-\delta}_{\alpha^\vee} = \mathrm{PT}_\mathcal{X}(\alpha)$. Hence, for δ positive (negative) enough, you have $\mathrm{PT}(\mathcal{X})_\beta$.

The full argument will appear in [BCR].

5.3 The crepant resolution conjecture

We prove the reinterpretation of the crepant resolution conjecture for Donaldson–Thomas invariants in Theorem 5.1.5. For the convenience of the reader, we repeat it here.

Theorem 5.3.1. Let $\mathcal{X}$ be a three-dimensional Calabi–Yau orbifold satisfying the hard Lefschetz condition, and assume that its coarse moduli space $\pi\colon \mathcal{X} \to \mathrm{X}$ is projective. Let $f\colon \mathrm{Y} \to \mathrm{X}$ denote its natural crepant resolution. For each multi-regular curve class $\beta \in \mathrm{N}_{\mathrm{mr}}(\mathcal{X})$ there exists a rational function $f_\beta(q)$ and an element $c_\beta \in \mathrm{N}_0(\mathcal{X})$ such that

1. the expansion of $f_\beta(q)$ with respect to $\mathrm{L}_{\mathrm{deg}}$ is the series $\mathrm{PT}(\mathcal{X})_\beta$,
2. the expansion of $f_\beta(q)$ with respect to some $\mathrm{L}_{\gamma'}$ is the series $\mathrm{PT}_f(\mathrm{Y}/\mathrm{X})_\beta$,
3. the poles of $f_\beta(q)$ lie on the locus the locus $\{q^{2\beta' \cdot \mathrm{A}} - 1 = 0\}$, where $\beta' \leq \beta$ and $\mathrm{A} \in \mathrm{Amp}(\mathrm{X})$, and potentially at $q = 0$.
4. the function $f_\beta(q)$ has the symmetry $f_\beta(q) = q^{c_\beta} f_\beta(q^\vee)$ where the isomorphism $(-)^\vee\colon \mathrm{N}_0(\mathcal{X}) \to \mathrm{N}_0(\mathcal{X})$ is induced by the shifted derived dual $\mathbf{D}$.

Proof of parts 1, 3, and 4. The existence of the rational function $f_\beta(q)$ and the element $c_\beta \in \mathrm{N}_0(\mathcal{X})$, and the first and third claim follow from Theorem 5.2.19 and its Corollary 5.2.20. The fourth claim is conjectural, see Conjecture 5.2.21. □

In this section, we prove the second and final claim, thus completing the proof of the crepant resolution conjecture. We follow the strategy of *Step 3* outlined in section 5.1.

First, we define a family of stability conditions that interpolates between stable pair invariants on $\mathcal{X}$ and Bryan–Steinberg invariants on the resolution $f\colon \mathrm{Y} \to \mathrm{X}$. It depends on two parameters $\gamma \in \mathbf{R}_{>0}$ and $\eta \in \mathbf{R}$. We locate the walls and show that (γ, η)-pairs for $0 < \gamma \ll 1$ are stable pairs. We also prove certain finiteness results allowing us to define curve-counting invariants. Second, we show that their generating series is an expansion of a rational function and that this function is unchanged under a γ-wall crossing. And third, we prove that (γ, η)-pairs for $\gamma \gg 0$ are Bryan–Steinberg pairs.

Remark 5.3.2. We emphasize that $\mathcal{X}$ satisfies the hard Lefschetz property in the remainder of this chapter. Moreover, we remind the reader that we continue to use the notation and conventions summarised below Remark 5.2.1.

5.3.1 Zeta-stability and boundedness

Recall the McKay equivalence $\Phi\colon \mathrm{D}(\mathrm{Y}) \to \mathrm{D}(\mathcal{X})$ of section 2.4 that sends $\Phi(\mathcal{O}_\mathrm{Y}) = \mathcal{O}_\mathcal{X}$. It induces an identification of the numerical Grothendieck groups $\phi\colon \mathrm{N}(\mathrm{Y}) \to \mathrm{N}(\mathcal{X})$.

Recall that it sends exceptional classes on Y to zero-dimensional classes on $\mathcal{X}$, i.e., $\phi(\mathrm{N}_{\mathrm{exc}}(\mathrm{Y})) = \mathrm{N}_0(\mathcal{X})$, and that $f\colon \mathrm{Y} \to \mathrm{X}$ satisfies $\dim f^{-1}(x) \le 1$ for all $x \in \mathrm{X}$.

Let $(\gamma, \eta) \in \mathbf{R}_{\ge 0} \times \mathbf{R}$ and fix an ample class ω on Y. Define the linear function

$$\mathrm{L}_\gamma\colon \mathrm{N}_0(\mathcal{X})_{\mathbf{R}} \to \mathbf{R}, \quad \mathrm{L}_\gamma(c) = \deg(c) + \gamma|\operatorname{ch}_2(\Psi(c)) \cdot \omega|_{\mathrm{Y}}, \tag{5.3.1}$$

where we write $\Psi = \Phi^{-1}$, where $|-|_{\mathrm{Y}}$ denotes the degree on Y of a class in $\mathrm{N}_0(\mathrm{Y})$, and where ch_2 denotes the second Chern character on Y. If c is the class of a zero-dimensional sheaf supported on a stacky point of $\mathcal{X}$, then $\mathrm{L}_\gamma(c)$ is a measure of the (non-)triviality of its representation type. We explain this linear function in an

Example 5.3.3. Let $\mathcal{X}$ be the total space $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to [\mathbf{P}^1/\mathbf{Z}_2]$ on the stacky local projective line discussed in section 3.1. Its coarse moduli space is the trivial $\mathbf{P}^1$-family of A_1-surface singularities and its crepant resolution $f\colon \mathrm{Y} \to \mathrm{X}$ is local $\mathbf{P}^1 \times \mathbf{P}^1$.

Let $x \in \mathcal{X}$ denote a non-stacky point, let $p \in \mathrm{X}$ denote a point in the singular locus, and let $f_p = f^{-1}(p) \subset \mathrm{Y}$ denotes its fibre in Y. Write $\mathcal{O}_p^+$ for the skyscraper sheaf with trivial $\mathbf{Z}_2$-equivariant structure, and $\mathcal{O}_p^-$ for the one with non-trivial structure. By Lemma 3.1.2, we have

$$\Phi(\mathcal{O}_{f_p}(-2)[1]) = \mathcal{O}_p^+ \quad \text{and} \quad \Phi(\mathcal{O}_{f_p}(-1)) = \mathcal{O}_p^-. \tag{5.3.2}$$

Hence $\mathrm{L}_\gamma([\mathcal{O}_p^+]) = \deg(\mathcal{O}_p^+) - \gamma(f_p \cdot \omega)$ whereas $\mathrm{L}_\gamma([\mathcal{O}_p^-]) = \deg(\mathcal{O}_p^-) + \gamma(f_p \cdot \omega)$. Note that the class $[\mathcal{O}_x] = [\mathcal{O}_p^+] + [\mathcal{O}_p^-] = [\mathcal{O}_p \otimes \mathbf{C}\mathbf{Z}_2]$ is multi-regular. We have $\mathrm{L}_\gamma([\mathcal{O}_x]) = \deg(\mathcal{O}_x)$, so that L_γ reduces to the degree function $\mathrm{L}_{\deg}$ for non-stacky classes.

We now introduce a two-parameter family of stability conditions.

Definition 5.3.4. Let $(\gamma, \eta) \in \mathbf{R}_{\ge 0} \times \mathbf{R}$. Define $\zeta_{\gamma,\eta}\colon \mathrm{N}_{\le 1}(\mathcal{X}) \to (-\infty, +\infty]^2$ as follows. If α is a one-dimensional class, then

$$\zeta_{\gamma,\eta}(\alpha) = (z_\gamma(\alpha), w_\eta(\alpha)) = \left(-\frac{\mathrm{L}_\gamma(\mathrm{A} \cdot \alpha)}{\mathrm{L}_{\deg}(\mathrm{A} \cdot \alpha)}, \nu(\alpha) + \eta \right) \in (-\infty, +\infty]^2. \tag{5.3.3}$$

If $\alpha \in \mathrm{N}_0(\mathcal{X})$ then we set $\zeta_{\gamma,\eta}(\alpha) = (\infty, \infty)$. Here $(\infty, \infty]^2$ is a totally ordered set via the lexicographical ordering, so $(a, b) \le (a', b')$ if $a < a'$, or if $a = a'$ and $b \le b'$.

We first show that $\zeta_{\gamma,\eta}$ defines a stability condition on $\operatorname{Coh}_{\le 1}(\mathcal{X})$ in the sense of Definition 2.1.23. Note that for sheaves $\mathrm{E}, \mathrm{F} \in \operatorname{Coh}_{\le 1}(\mathcal{X})$, the internal ordering of $\zeta_{\gamma,\eta}(\mathrm{E})$ and $\zeta_{\gamma,\eta}(\mathrm{F})$ is independent of both γ and η. Thus if $\zeta_{\gamma,\eta}$ is a stability condition for *a* choice of (γ, η), then it is so for *all* choices.

Lemma 5.3.5. The category $\operatorname{Coh}_{\le 1}(\mathcal{X})$ is $\zeta_{\gamma,\eta}$-artinian for all $(\gamma, \eta) \in \mathbf{R}_{>0} \times \mathbf{R}$.

Proof. Let $\mathrm{F} \in \mathrm{Coh}_{\leq 1}(\mathcal{X})$, and assume for a contradiction that $\mathrm{F} = \mathrm{F}_0 \supset \mathrm{F}_1 \supset \cdots$ is an infinite chain of subobjects with $\zeta_{\gamma,\eta}(\mathrm{F}_i) \geq \zeta_{\gamma,\eta}(\mathrm{F}_{i-1})$ for all i. Now, as $\beta_{\mathrm{F}_i} \leq \beta_{\mathrm{F}_{i-1}}$ and the set $\{\beta \mid 0 \leq \beta \leq \beta_{\mathrm{F}}\}$ is finite, we may reduce to the case where $\beta_{\mathrm{F}_i} = \beta_{\mathrm{F}}$ for all i. But then $\zeta_{\gamma,\eta}(\mathrm{F}_i) \geq \zeta_{\gamma,\eta}(\mathrm{F}_{i-1})$ implies that $\nu(\mathrm{F}_i) \geq \nu(\mathrm{F}_{i-1})$ for all i, which is impossible by the existence of Harder–Narasimhan filtrations for ν as in 2.1.41 ☐

Corollary 5.3.6. The function $\zeta_{\gamma,\eta}$ defines a stability condition on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ for all $(\gamma, \eta) \in \mathbf{R}_{>0} \times \mathbf{R}$.

Proof. It is easy to see that $\zeta_{\gamma,\eta}$ satisfies the see-saw property. ☐

Similarly to the case of Nironi-stability, $\zeta_{\gamma,\eta}$-stability induces a torsion pair on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ by collapsing its Harder–Narasimhan filtration:

$$\begin{aligned} \mathtt{T}_{\zeta_{\gamma,\eta}} &:= \{\mathrm{T} \in \mathrm{Coh}_{\leq 1}(\mathcal{X}) \,|\, \mathrm{T} \twoheadrightarrow \mathrm{Q} \neq 0 \Rightarrow \zeta_{\gamma,\eta}(\mathrm{Q}) \geq (0,0)\} \\ \mathtt{F}_{\zeta_{\gamma,\eta}} &:= \{\mathrm{F} \in \mathrm{Coh}_{\leq 1}(\mathcal{X}) \,|\, 0 \neq \mathrm{S} \hookrightarrow \mathrm{F} \Rightarrow \zeta_{\gamma,\eta}(\mathrm{F}) < (0,0)\}. \end{aligned} \tag{5.3.4}$$

We have a two-parameter family of numerical torsion pairs $(\mathtt{T}_{\zeta_{\gamma,\eta}}, \mathtt{F}_{\zeta_{\gamma,\eta}})$ on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$.

Definition 5.3.7. We write $\mathtt{P}_{\zeta_{\gamma,\eta}} \subset \mathtt{A}$ for the category of $(\mathtt{T}_{\zeta_{\gamma,\eta}}, \mathtt{F}_{\zeta_{\gamma,\eta}})$-pairs.

Remark 5.3.8. Varying the parameters (γ, η) only changes the $\zeta_{\gamma,\eta}$-slope of an object in $\mathrm{Coh}_{\leq 1}(\mathcal{X})$, not whether or not it is $\zeta_{\gamma,\eta}$-(semi)stable. However, varying (γ, η) *does* affect the notion of $(\mathtt{T}_{\zeta_{\gamma,\eta}}, \mathtt{F}_{\zeta_{\gamma,\eta}})$-pair and later, hence, the associated counting invariants.

We prove that the torsion pair $(\mathtt{T}_{\zeta_{\gamma,\eta}}, \mathtt{F}_{\zeta_{\gamma,\eta}})$ is open by giving a characterisation of membership of the categories $\mathtt{T}_{\zeta_{\gamma,\eta}}$ and $\mathtt{F}_{\zeta_{\gamma,\eta}}$ that is open in flat families.

To do so, we first need two definitions. We single out the first part of ζ-stability.

Definition 5.3.9. Define a family of slope functions $\theta_\gamma\colon \mathrm{N}_0(\mathcal{X}) \to \mathbf{R}$ by setting

$$\theta_\gamma(c) = -1 - \gamma \frac{|\,\mathrm{ch}_2(\Psi(c)) \cdot \omega|_{\mathrm{Y}}}{\deg(c)} \tag{5.3.5}$$

if $\deg(c) \neq 0$ and $\theta_\gamma(c) = \infty$ otherwise.

The θ_γ satisfy the see-saw property. Hence, they define stability conditions on $\mathrm{N}_0(\mathcal{X})$ in the sense of Definition 2.1.23 because the category $\mathrm{Coh}_0(\mathcal{X})$ is artinian. In particular, objects in $\mathrm{Coh}_0(\mathcal{X})$ have θ_γ-Harder–Narasimhan filtrations.

Thus, we may define a torsion pair $(\mathtt{T}_{\theta_\gamma}, \mathtt{F}_{\theta_\gamma})$ on $\mathrm{Coh}_0(\mathcal{X})$ by setting

$$\begin{aligned} \mathtt{T}_{\theta_\gamma} &:= \{\mathrm{T} \in \mathrm{Coh}_0(\mathcal{X}) \mid \mathrm{T} \twoheadrightarrow \mathrm{Q} \neq 0 \Rightarrow \theta_\gamma(\mathrm{Q}) \geq 0\} \\ \mathtt{F}_{\theta_\gamma} &:= \{\mathrm{F} \in \mathrm{Coh}_0(\mathcal{X}) \mid 0 \neq \mathrm{S} \hookrightarrow \mathrm{F} \Rightarrow \theta_\gamma(\mathrm{S}) < 0\}. \end{aligned} \tag{5.3.6}$$

Note that $\mathtt{T}_{\theta_\gamma}$ is closed under extension and quotients, so we may apply Lemma 2.1.17.

Lemma 5.3.10. The torsion pair $(\mathtt{T}_{\theta_\gamma}, \mathtt{F}_{\theta_\gamma})$ is open.

Proof. We must show that the substacks $\underline{\mathtt{T}}_{\theta_\gamma}$ and $\underline{\mathtt{F}}_{\theta_\gamma}$, parametrising objects in $\mathtt{T}_{\theta_\gamma}$ and $\mathtt{F}_{\theta_\gamma}$ respectively, are open in $\underline{\mathrm{Coh}}_{\mathcal{X},0}$. This follows from the arguments of [HL10, Thm. 2.3.1], since there are at most finitely many classes of potentially destabilising quotients. This in turn follows because the set $\{0 \le c' \le c \mid c' \in \mathrm{N}_0(\mathcal{X})\}$ is finite for every $c \in \mathrm{N}_0(\mathcal{X})$ by the fact that $\mathrm{Coh}_0(\mathcal{X})$ is artinian. ☐

Definition 5.3.11. Let $\mathrm{E} \in \mathrm{Coh}_{\le 1}(\mathcal{X})$ be a sheaf, and let $n \in \mathbf{Z}_{>0}$. We say that a pencil $\mathrm{L} = \mathbf{P}^1 \subset |n\mathrm{A}|$ is a *good pencil for* E if the following conditions hold:

1. the base locus of L intersects neither supp(E) nor the singular locus of X, and
2. no one-dimensional component of supp(E) is contained in any member of L.

For a point $p \in \mathrm{L}$ we denote the associated divisor substack by $\mathrm{D}_p \hookrightarrow \mathcal{X}$. Let $\mathcal{X}_\mathrm{L}$ denote the blow-up of $\mathcal{X}$ in the base locus of L, so that we have a natural morphism $b\colon \mathcal{X}_\mathrm{L} \to \mathrm{L}$.

Remark 5.3.12. Since X is projective, we may embed it in some projective space. There exists a good pencil for every $\mathrm{E} \in \mathrm{Coh}_{\le 1}(\mathcal{X})$ by Bertini's Theorem.

We now state an alternative characterisation of membership of $\mathtt{T}_{\zeta_\gamma,\eta}$ and $\mathtt{F}_{\zeta_\gamma,\eta}$.

Lemma 5.3.13. Let $\mathrm{E} \in \mathrm{Coh}_{\le 1}(\mathcal{X})$, and let L be a good pencil for E. Then $\mathrm{E} \in \mathtt{T}_{\zeta_\gamma,\eta}$ if and only if it satisfies conditions T1 and T2.

(T1) There exists a $p \in \mathrm{L}$ such that the restriction $\mathrm{E}|_{\mathrm{D}_p}$ lies in $\mathtt{T}_{\theta_\gamma}$.

(T2) The sheaf E admits no quotient sheaf Q with

$$\frac{\mathrm{L}_\gamma(\mathrm{A}\cdot\beta_\mathrm{Q})}{\mathrm{L}_{\deg}(\mathrm{A}\cdot\beta_\mathrm{Q})} = 0 \tag{5.3.7}$$

and $\nu(\mathrm{F}) < -\eta$.

We have $\mathrm{E} \in \mathtt{F}_{\zeta_\gamma,\eta}$ if and only if it satisfies conditions F0, F1, and F2.

(F0) The sheaf E is pure.

(F1) There exists a $p \in \mathrm{L}$ such that the restriction $\mathrm{E}|_{\mathrm{D}_p}$ lies in $\mathtt{F}_{\theta_\gamma}$

(F2) The sheaf E admits no subsheaf S with

$$\frac{\mathrm{L}_\gamma(\mathrm{A}\cdot\beta_\mathrm{S})}{\mathrm{L}_{\deg}(\mathrm{A}\cdot\beta_\mathrm{S})} = 0 \tag{5.3.8}$$

and $\nu(\mathrm{F}) \ge -\eta$.

Proof. We only treat the characterisation of membership of $\mathtt{T}_{\zeta_{\gamma,\eta}}$ since the arguments for $\mathtt{F}_{\zeta_{\gamma,\eta}}$ are similar. It is easy to see that the proof is complete once we show that E violating condition (T1) is equivalent to the existence of a quotient $\mathrm{E} \twoheadrightarrow \mathrm{Q}$ such that

$$\frac{-\mathrm{L}_\gamma(\mathrm{A}\cdot\beta_{\mathrm{Q}})}{\mathrm{L}_{\deg}(\mathrm{A}\cdot\beta_{\mathrm{Q}})} < 0. \tag{5.3.9}$$

First assume that such a quotient $\mathrm{E} \twoheadrightarrow \mathrm{Q}$ exists. For a general point $p \in \mathrm{L}$, its restriction $\mathrm{Q}|_{\mathrm{D}_p}$ is a quotient of $\mathrm{E}|_{\mathrm{D}_p}$ since L is a good pencil for E. This shows that $\mathrm{E}|_{\mathrm{D}_p} \notin \mathtt{T}_{\theta_\gamma}$.

Conversely, suppose that condition (T1) does not hold. Since the support of E is disjoint from the base locus of L, we may think of E as a sheaf on the blow-up $b\colon \mathcal{X}_{\mathrm{L}} \to \mathrm{L}$. There exists an open subset $\mathrm{U} \subseteq \mathrm{L}$ such that $\mathrm{E}|_{\mathrm{U}}$ is flat over U. Let $p \in \mathrm{U}$, and consider the θ_γ-HN filtration of the sheaf $\mathrm{E}|_{\mathrm{D}_p}$. It is of the form $0 = \mathrm{F}_0 \subset \mathrm{F}_1 \subset \ldots \subset \mathrm{F}_n = \mathrm{E}|_{\mathrm{D}_p}$. Let $c_p \in \mathrm{N}_0(\mathcal{X}_{\mathrm{L}})$ denote the class of $\mathrm{F}_n/\mathrm{F}_{n-1}$. By [HL10, Thm. 2.3.2], the top HN-factors $\mathrm{F}_n/\mathrm{F}_{n-1}$ form a flat family after potentially shrinking U. Thus there exists a class c such that $c_p = c$ for all $p \in \mathrm{U}$. By our assumption, we have $\theta_\gamma(c) < 0$.

The relative Quot scheme $\mathrm{Quot}^c(\mathrm{E}|_{\mathrm{U}}/\mathrm{U})$ is proper over U. Because the θ_γ-HN filtration is unique, for a general $p \in \mathrm{U}$ there is a unique closed point in $\mathrm{Quot}^c(\mathrm{E}|_{\mathrm{D}_p})$, and so there exists a unique section $\mathrm{U} \to \mathrm{Quot}^c(\mathrm{E}|_{\mathrm{U}}/\mathrm{U})$. Let $\mathrm{E}|_{\mathrm{U}} \twoheadrightarrow \mathrm{Q}'$ be the associated surjection, let $j\colon b^{-1}(\mathrm{U}) \hookrightarrow \mathcal{X}_{\mathrm{L}}$ be the inclusion, and let Q be the image of $\mathrm{E} \to j_*j^*\mathrm{E} \to j_*\mathrm{Q}'$. Then the surjection $\mathrm{E} \twoheadrightarrow \mathrm{Q}$ shows that $\mathrm{E} \notin \mathtt{T}_{\zeta_{\gamma,\eta}}$. □

We now prove that the torsion pair $(\mathtt{T}_{\zeta_{\gamma,\eta}}, \mathtt{F}_{\zeta_{\gamma,\eta}})$ is open.

Lemma 5.3.14. Conditions (T1), (F0), and (F1) are open in flat families in $\mathrm{Coh}_{\le 1}(\mathcal{X})$.

Proof. Let S be the base scheme of a flat family of sheaves in $\mathrm{Coh}_{\le 1}(\mathcal{X})$, and let E_s be the sheaf corresponding to some point $s \in \mathrm{S}$. There exists a good pencil $\mathrm{L} \subset |n\mathrm{A}|$ for E_s. Suppose that E_s satisfies condition (T1), and let $p \in \mathrm{L}$ be a point for which the restriction $(\mathrm{E}_s)|_{\mathrm{D}_p}$ lies in $\mathtt{T}_{\theta_\gamma}$. Picking a suitable open neighbourhood $s \in \mathrm{U} \subset \mathrm{S}$, the pencil L remains good for all sheaves in the neighbourhood. The value of θ_γ is constant in flat families since the numerical class is constant, so $(\mathrm{E}_u)|_{\mathrm{D}_p}$ lies in $\mathtt{T}_{\theta_\gamma}$ for all $u \in \mathrm{U}$.

Openness of condition (F1) follows by the same argument, whereas openness of condition (F0) follows from the proof of Lemma 4.2.7. □

Lemma 5.3.15. Conditions (T2) and (F2) are open in flat families in $\mathrm{Coh}_{\le 1}(\mathcal{X})$.

Proof. This follows by openness of Nironi-stability in flat families, i.e., by openness of the torsion pair $(\mathtt{T}_\delta, \mathtt{F}_\delta)$ as proved in part 1 of Proposition 5.2.4. □

Corollary 5.3.16. The torsion pair $(\mathtt{T}_{\zeta_{\gamma,\eta}}, \mathtt{F}_{\zeta_{\gamma,\eta}})$ is open for all $(\gamma,\eta) \in \mathbf{R}_{>0} \times \mathbf{R}$.

Proof. The torsion pair is open by Lemmas 5.3.13, 5.3.14, and 5.3.15. $\square$

Corollary 5.3.17. The moduli stack of $(\mathtt{T}_{\zeta_{\gamma,\eta}}, \mathtt{F}_{\zeta_{\gamma,\eta}})$-pairs defines an open substack $\underline{\mathrm{P}}_{\zeta_{\gamma,\eta}} \subset \mathfrak{Mum}_{\mathcal{X}}$. In particular, it is an algebraic stack locally of finite type.

Proof. Since $\mathrm{Coh}_0(\mathcal{X}) \subset \mathtt{T}_{\zeta_{\gamma,\eta}}$ for all $(\gamma,\eta) \in \mathbf{R}_{>0} \times \mathbf{R}$, this follows by Corollary 4.2.13 $\square$

Next we prove boundedness properties of the moduli stack of $(\mathtt{T}_{\zeta_{\gamma,\eta}}, \mathtt{F}_{\zeta_{\gamma,\eta}})$-pairs. Consider the following 'limit' subcategories of $\mathrm{Coh}_{\le 1}(\mathcal{X})$.

$$\mathtt{T}_{\zeta_{\gamma,-\infty}} = \bigcup_{\eta \in \mathbf{R}} \mathtt{T}_{\zeta_{\gamma,\eta}} \quad \text{and} \quad \mathtt{F}_{\zeta_{\gamma,\infty}} = \bigcup_{\eta \in \mathbf{R}} \mathtt{F}_{\zeta_{\gamma,\eta}}. \tag{5.3.10}$$

We have the following auxiliary boundedness results.

Lemma 5.3.18. Let $(\gamma,\eta_i) \in \mathbf{R}_{>0} \times \mathbf{R}$ for $i = 1,2$ and let $\beta \in \mathrm{N}_1(\mathcal{X})$. The sets

$$\{c_{\mathrm{F}} \in \mathrm{N}_0(\mathcal{X}) \mid \exists \mathrm{F} \in \mathtt{T}_{\eta_1} \cap \mathtt{F}_{\zeta_{\gamma,\eta_2}} \text{ with } \beta_{\mathrm{F}} \le \beta\},$$
$$\{c_{\mathrm{F}} \in \mathrm{N}_0(\mathcal{X}) \mid \exists \mathrm{F} \in \mathtt{F}_{\eta_1} \cap \mathtt{T}_{\zeta_{\gamma,\eta_2}} \text{ with } \beta_{\mathrm{F}} \le \beta\}$$

are each L_γ-bounded. The sets

$$\{c_{\mathrm{F}} \in \mathrm{N}_0(\mathcal{X}) \mid \exists \mathrm{F} \in \mathtt{T}_{\eta} \cap \mathtt{F}_{\zeta_{\gamma,\infty}} \text{ with } \beta_{\mathrm{F}} \le \beta\},$$
$$\{c_{\mathrm{F}} \in \mathrm{N}_0(\mathcal{X}) \mid \exists \mathrm{F} \in \mathtt{F}_{\eta} \cap \mathtt{T}_{\zeta_{\gamma,-\infty}} \text{ with } \beta_{\mathrm{F}} \le \beta\}$$

are each weakly L_γ-bounded.

Recall that $(\mathtt{T}_\delta, \mathtt{F}_\delta)$ is the torsion pair obtained by collapsing the HN filtration of Nironi-stability at slope $\delta \in \mathbf{R}$. It is defined directly above Definition 5.2.2

Proof. We only prove the claim for the first and third set, as the other two sets can be dealt with by a similar argument.

We write S for the first set. First assume that $\eta_1 = \eta$ and $\eta_2 = -\eta$. We have to prove that $\mathrm{S} \cap \{c \in \mathrm{N}_0(\mathcal{X}) \mid \mathrm{L}_\gamma(c) \le \mathrm{M}\}$ is a finite set for all $\mathrm{M} \in \mathbf{R}$. Let $\mathrm{F} \in \mathtt{T}_\eta \cap \mathtt{F}_{\zeta_{\gamma,-\eta}}$ be such that $\beta_{\mathrm{F}} \le \beta$. Note that $\nu_{\min}(\mathrm{F}) \ge \eta$ and that F is pure one-dimensional. The sheaf F admits a Harder–Narasimhan filtration with respect to ν. For any $k \in \mathbf{Z}_{>0}$, we may collapse this filtration into k parts wherein the filtration quotients

$$\mathrm{F}_{[\eta,\eta+1)}, \mathrm{F}_{[\eta+1,\eta+2)}, \ldots, \mathrm{F}_{[\eta+k,\eta+k+1)}, \tag{5.3.11}$$

satisfy $\mathrm{F}_{\mathrm{I}} \in \mathcal{M}_\nu^{\mathrm{ss}}(\mathrm{I})$; of course, some F_{I} may be zero.

Let $\mathrm{S}_m \subseteq \mathrm{S}$ denote the set of F with at most m non-vanishing pieces in this collapsed filtration, and let $\mathrm{S}'_m \subseteq \mathrm{S}_m$ be the subset of those for which $\mathrm{F}_{[\eta,\eta+1)} \neq 0$. Note that S'_1 is a subset of

$$\mathrm{R} = \{\mathrm{F} \in \mathrm{Coh}_1(\mathcal{X}) \mid \beta_\mathrm{F} \leq \beta,\, \mathrm{F} \in \mathcal{M}^{\mathrm{ss}}_\nu([\eta, \eta+1))\}. \tag{5.3.12}$$

Hence $c(\mathrm{S}'_1) := \{c_\mathrm{F} \in \mathrm{N}_0(\mathcal{X}) \mid \mathrm{F} \in \mathrm{S}'_1\} \subseteq c(\mathrm{R})$ is finite by Theorem 2.1.47.

Since $\mathrm{F} \in \mathtt{F}_{\zeta_{\gamma,-\eta}}$ we have $\zeta_{\gamma,-\eta}(\mathrm{F}) < (0,0)$. But the condition $\mathrm{L}_\gamma(\beta_\mathrm{F}\cdot \mathrm{A}) = 0$ and $\nu_{\min}(\mathrm{F}) - \eta < 0$ contradicts $\mathrm{F} \in \mathtt{T}_\eta$, since $\mathrm{F} \in \mathtt{T}_\eta$ means $\nu_{\min}(\mathrm{F}) \geq \eta$ by definition. Thus $\zeta_{\gamma,-\eta}(\mathrm{F}) < (0,0)$ implies $\mathrm{L}_\gamma(\beta_\mathrm{F}\cdot\mathrm{A}) > 0$ for all $\mathrm{F} \in \mathrm{S}$. Moreover, note that the set $\mathrm{Q} = \mathbf{Z}_{>0}\{\beta'\cdot\mathrm{A} \mid \beta' \leq \beta \text{ and } \mathrm{L}_\gamma(\beta'\cdot\mathrm{A}) > 0\}$ is L_γ-bounded. It follows that twisting by A implies $c(\mathrm{S}_1) \subseteq c(\mathrm{S}'_1) + \mathrm{Q}$ because $\nu(\mathrm{F}\otimes\mathrm{A}) = \nu(\mathrm{F}) + 1$. Thus $c(\mathrm{S}_1)$ is L_γ-bounded.

Now take $\mathrm{F} \in \mathrm{S}_m$. Decomposing F into $\mathrm{F}_{[\eta,\eta+1)}$ and $\mathrm{F}_{[\eta+1,\infty)}$, we have $\mathrm{F}_{[\eta,\eta+1)} \in \mathrm{R}$ and $\mathrm{F}_{[\eta+1,\infty)} \in \mathrm{S}_{m-1}$. This shows that $c(\mathrm{S}'_m) \subseteq c(\mathrm{S}_{m-1}) + c(\mathrm{R})$. Moreover, since $c(\mathrm{S}_m) \subseteq c(\mathrm{S}'_m) + \mathrm{Q}$, we deduce that $c(\mathrm{S}_m) \subseteq c(\mathrm{S}_{m-1}) + \mathrm{Q} + \mathrm{R}$. By induction, we have that $c(\mathrm{S}_m)$ is L_γ-bounded. For $m \geq d_\beta$ we have $\mathrm{S}_m = \mathrm{S}$, and so the claim follows.

Now let $\eta_1, \eta_2 \in \mathbf{R}$ be arbitrary. Without loss of generality we may assume $\eta_1 < -\eta_2$ for otherwise we reduce to the known claim. Any $\mathrm{F} \in \mathtt{T}_{\eta_1} \cap \mathtt{F}_{\zeta_{\gamma,\eta_2}}$ admits a ν-HN filtration

$$0 \to \mathrm{F}_{\geq -\eta_2} \to \mathrm{F} \to \mathrm{F}_{[\eta_1,-\eta_2)} \to 0 \tag{5.3.13}$$

where the subscripts indicate the allowed ν-slopes. Since the curve class of $\mathrm{F}_{[\eta_1,-\eta_2)}$ is $\leq \beta$, it can only produce finitely many classes in $\mathrm{N}_0(\mathcal{X})$ by Theorem 2.1.47. Since $\mathtt{F}_{\zeta_{\gamma,\eta_2}}$ is closed under subobjects, we have $\mathrm{F}_{\geq -\eta_2} \in \mathtt{T}_{-\eta_2} \cap \mathtt{F}_{\zeta_{\gamma,\eta_2}}$. Thus by the case $(-\eta_2, \eta_2)$ the set of possible classes for $\mathrm{F}_{\geq-\eta_2}$ is L_γ-bounded. The sum of an L_γ-bounded set with a finite set is again L_γ-bounded. This completes the claim for the first mentioned set.

For the third mentioned set, define $\mathrm{Q} = \mathbf{Z}_{>0}\{\beta'\cdot\mathrm{A} \mid \beta' \leq \beta \text{ and } \mathrm{L}_\gamma(\beta\cdot\mathrm{A}) \geq 0\}$. This set is only weakly L_γ-bounded. We conclude by the same argument. ☐

Proposition 5.3.19. For any $(\gamma,\eta) \in \mathbf{R}_{>0} \times \mathbf{R}$, the set

$$\{c \in \mathrm{N}_0(\mathrm{X}) \mid \underline{\mathtt{P}}_{\zeta_{\gamma,\eta}}(\beta, c) \neq \varnothing\} \tag{5.3.14}$$

is L_γ-bounded. Moreover, the stack $\underline{\mathtt{P}}_{\zeta_{\gamma,\eta}}(\beta, c)$ is of finite type.

Proof. Let $\mathrm{E} \in \mathtt{P}_{\zeta_{\gamma,\eta}}$ be a $(\mathtt{T}_{\zeta_{\gamma,\eta}}, \mathtt{F}_{\zeta_{\gamma,\eta}})$-pair. By Proposition 4.1.24, it has a three-term filtration induced by the torsion triple $(\mathtt{T}_\eta, \mathrm{V}(\mathtt{T}_\eta, \mathtt{F}_\eta), \mathtt{F}_\eta)$ on $\mathtt{A}$ that is associated to the torsion pair $(\mathtt{T}_\eta, \mathtt{F}_\eta)$ of Nironi stability. Concretely, this means that we have objects

$$\mathrm{E}_{\geq\eta} \hookrightarrow \mathrm{E}, \quad \text{and} \quad \mathrm{E} \twoheadrightarrow \mathrm{E}_{<\eta}, \tag{5.3.15}$$

where the first filtration quotient is $\mathrm{E}_{\geq\eta} \in \mathtt{T}_\eta$, the second filtration quotient is the η-pair

$\mathrm{E}_{\mathtt{P}_\eta} := \ker(\mathrm{E} \twoheadrightarrow \mathrm{E}_{<\eta})/\mathrm{E}_{\geq\eta} \in \mathtt{P}_\eta$, and the third filtration quotient is $\mathrm{E}_{<\eta} \in \mathtt{F}_\eta$.

Since $\mathrm{E} \in \mathtt{P}_{\zeta_{\gamma,\eta}}$ we have $\mathrm{E}_{\geq\eta} \in \mathtt{F}_{\gamma,\eta}$ and $\mathrm{E}_{<\eta} \in \mathtt{T}_{\gamma,\eta}$. By Lemma 5.3.18, it follows that the sets of possible values for $c(\mathrm{E}_{\geq\eta})$ and $c(\mathrm{E}_{<\eta})$ are both L_γ-bounded. Moreover, the set of possible values for $c(\mathrm{E}_{\mathtt{P}_\eta}) \in \mathtt{P}_\eta$ is finite, by part 1 of Lemma 5.2.11. Thus, the set in equation (5.3.14) is L_γ-bounded.

Given a fixed $(\beta, c) \in \mathrm{N}_{\leq 1}(\mathcal{X})$, there are finitely many possible curve classes for $\mathrm{E}_{\geq\eta}, \mathrm{E}_{\mathtt{P}_\eta}$, and $\mathrm{E}_{<\eta}$. The above argument shows that there are only finitely many possible values for the class of $\mathrm{E}_{\mathtt{P}_\eta}$. Since $c - c(\mathrm{E}_{\mathtt{P}_\eta}) = c(\mathrm{E}_{\geq\eta}) + c(\mathrm{E}_{<\eta})$ and the last two classes lie in an L_γ-bounded set, we conclude that only *finitely* many classes for $\mathrm{E}_{\geq\eta}$ and $\mathrm{E}_{<\eta}$ can appear. Moreover, their Nironi slopes are bounded. On the one hand, we have

$$\nu_{\min}(\mathrm{E}_{\geq\eta}) \geq \eta \quad \text{and} \quad \nu_{\max}(\mathrm{E}_{<\eta}) < \eta. \tag{5.3.16}$$

Depending on the sign of η, the other two bounds are different:

1. If $\eta \leq 0$, the argument of Lemma 5.2.5 yields

$$\max\{0, \deg(\mathrm{E}_{\geq\eta}) - d_\beta\eta\} \geq \nu_{\max}(\mathrm{E}_{\geq\eta}) \quad \text{and} \quad \nu_{\min}(\mathrm{E}_{<\eta}) \geq \min\{0, \deg(\mathrm{E}_{<\eta})\}$$

2. Let $\eta \geq 0$, an argument similar to that of Lemma 5.2.5 yields

$$\max\{0, \deg(\mathrm{E}_{\geq\eta})\} \geq \nu_{\max}(\mathrm{E}_{\geq\eta}) \quad \text{and} \quad \nu_{\min}(\mathrm{E}_{<\eta}) \geq \min\{0, \deg(\mathrm{E}_{<\eta}) - d_\beta\eta\}.$$

where $\nu(\beta, c) = \deg(\beta, c)/d_\beta$. Note that the degree of a sheaf only depends on its numerical class and, by the previous argument, there are only finitely many possible classes for $\mathrm{E}_{\geq\eta}$ and $\mathrm{E}_{<\eta}$. It follows from Theorem 2.1.47 that the stack of such $\mathrm{E}_{\mathtt{F}_\eta}$ and $\mathrm{E}_{\mathtt{T}_\eta}$ is of finite type. Since the same is true for the stack of $\mathrm{E}_{\mathtt{P}_\eta}$ by part 2 of Proposition 5.2.11, it follows that the stack $\underline{\mathtt{P}}_{\zeta_{\gamma,\eta}}(\beta, c)$ is of finite type. ☐

Corollary 5.3.20. Let $(\gamma, \eta) \in \mathbf{R}_{>0} \times \mathbf{R}$. The category $\mathtt{P}_{\zeta_{\gamma,\eta}}$ of $(\mathtt{T}_{\zeta_{\gamma,\eta}}, \mathtt{F}_{\zeta_{\gamma,\eta}})$-pairs defines an element in the graded Hall algebra $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$ in the sense of Definition 2.3.30.

Proof. By the previous lemma, the stack $\underline{\mathtt{P}}_{\zeta_{\gamma,\eta}}(\beta, c)$ is of finite type for each numerical class $(\beta, c) \in \mathrm{N}_{\leq 1}(\mathcal{X})$. This completes the proof. ☐

Finally, recall that whether or not a sheaf in $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ is $\zeta_{\gamma,\eta}$-(semi)stable does not depend on $(\gamma, \eta) \in \mathbf{R}_{>0} \times \mathbf{R}$. However, varying (γ, η) alters their slopes. For simplicity, we suppress (γ, η). Let $\mathcal{M}^{\mathrm{ss}}_\zeta(a, b) \subset \mathrm{Coh}_{\leq 1}(\mathcal{X})$ denote the full subcategory of ζ-semistable sheaves of slopes $(a, b) \in \mathbf{R}^2$. We prove that its moduli are open and bounded.

Proposition 5.3.21. Let $(\beta, c) \in \mathrm{N}_{\leq 1}(\mathcal{X})$ be a class and let $(a, b) \in \mathbf{R}^2$ be a slope.

1. The moduli stack $\underline{\mathcal{M}}^{\mathrm{ss}}_{\zeta}(a,b) \subset \underline{\mathrm{Coh}}_{\leq 1,\mathcal{X}}$ is open and, in particular, it is an algebraic stack that is locally of finite type.

2. The category $\mathcal{M}^{\mathrm{ss}}_{\zeta}(a,b)$ is decompositionally finite in the sense of Definition 4.3.10.

Proof. For the first part, let S be a base scheme and let E be an S-flat family of sheaves in $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ of modified Hilbert polynomial p. Recall that a class $\alpha = (\beta, c) \in \mathrm{N}_{\leq 1}(\mathcal{X})$ determines a modified Hilbert polynomial via $p_\alpha(k) = d_\beta k + \deg(\beta, c)$. The set

$$\mathrm{D}_p := \left\{ p' \in \mathbf{Z}[x] \;\middle|\; \substack{\zeta_{\gamma,\eta}(p') < \zeta_{\gamma,\eta}(p),\ \exists s \in \mathrm{S} \text{ and} \\ \mathrm{E}_s \twoheadrightarrow \mathrm{Q} \text{ with Q pure and } p_{\mathrm{Q}} = p'} \right\} \tag{5.3.17}$$

is the set of modified Hilbert polynomials of objects that prevent a member E_s of the family E of being $\zeta_{\gamma,\eta}$-semistable; any such member satisfies $\zeta_{\gamma,\eta}(\mathrm{E}_s) = \zeta_{\gamma,\eta}(p)$. We have to show that the set $\mathrm{U} = \mathrm{S} \setminus \bigcup_{p' \in \mathrm{D}_p} \mathrm{im}(\pi_{p'}) \subset \mathrm{S}$ is open, where $\pi_p \colon \mathrm{Quot}_{\mathcal{X}}(\mathrm{E}/\mathrm{S}, p) \to \mathrm{S}$ is the relative Quot functor. Since the morphism π_p is projective by [Nir08, Thm. 4.20], it suffices to show that the set D_p is finite.

Recall that $\zeta_{\gamma,\eta}(p) = (z_\gamma(p), \nu(p) + \eta)$ by Definition 5.3.4. Let $s \in \mathrm{S}$ and let $\mathrm{E}_s \twoheadrightarrow \mathrm{Q}$ be a pure quotient. The inequality $\zeta_{\gamma,\eta}(p_{\mathrm{Q}}) < \zeta_{\gamma,\eta}(p)$ holds if either

(i) $z_\gamma(p_{\mathrm{Q}}) < z_\gamma(p)$, or

(ii) $z_\gamma(p_{\mathrm{Q}}) = z_\gamma(p)$ and $\nu(p_{\mathrm{Q}}) < \nu(p)$.

Case (i) occurs for finitely many linear polynomials $p_{\mathrm{Q}} \in \mathbf{Z}[x]$ because it only depends on the linear coefficient $d_{\beta_{\mathrm{Q}}}$, and the set $\{0 < \beta_{\mathrm{Q}} \leq \beta\}$ is finite. Case (ii) deals with the constant coefficient. By the Grothendieck Lemma for stacks [Nir08, Lem. 4.13], it also only occurs for finitely many p_{Q}. This completes the proof of the first part.

For the second part, note that the category $\mathcal{M}^{\mathrm{ss}}_{\zeta}(a,b)$ is closed under direct sums and direct summands by semistability. Suppose that $\alpha = (\beta, c) \in \mathrm{N}_{\leq 1}(\mathcal{X})$ decomposes as $\alpha = \alpha_1 + \alpha_2$ where the $\alpha_i = (\beta_i, c_i)$ are the classes of objects $\mathrm{E}_i \in \mathcal{M}^{\mathrm{ss}}_{\zeta}(a,b)$. If $\beta_i = 0$ then $\zeta_{\gamma,\eta}(\mathrm{E}_i) = (\infty, \infty) \neq (a,b)$. Thus $0 < \beta_1, \beta_2 < \beta$ and the set of such is finite by Lemma 2.1.38. We conclude that $\mathcal{M}^{\mathrm{ss}}_{\zeta}(a,b)$ is of finite length.

It remains to show that the moduli stack $\underline{\mathcal{M}}^{\mathrm{ss}}_{\zeta}(a,b)_\alpha$, parametrising ζ-semistable sheaves of slope (a,b) and class $\alpha = (\beta, c) \in \mathrm{N}_{\leq 1}(\mathcal{X})$, is of finite type. To do so, let E be such a sheaf and let $(\gamma, \eta) \in \mathbf{R}_{>0} \times \mathbf{R}$ be such that $\zeta_{\gamma,\eta}(\mathrm{E}) = (0,0)$. Decompose E with respect to the Nironi torsion pair

$$0 \to \mathrm{E}_{\geq \eta} \to \mathrm{E} \to \mathrm{E}_{<\eta} \to 0 \tag{5.3.18}$$

where $\mathrm{E}_{\geq\eta} \in \mathtt{T}_\eta$ and $\mathrm{E}_{<\eta} \in \mathtt{F}_\eta$. In particular, $\nu_{\min}(\mathrm{E}_{\geq\eta}) \geq \eta$ and $\nu_{\max}(\mathrm{E}_{<\eta}) \leq \eta$. By Lemma 2.1.38, there are finitely many possible curve classes for $\mathrm{E}_{\geq\eta}$ and $\mathrm{E}_{<\eta}$ since these satisfy $\beta' < \beta$. Thus, if we find an upper bound for $\nu_{\max}(\mathrm{E}_{\geq\eta})$ and a lower bound for $\nu_{\min}(\mathrm{E}_{<\eta})$, we may conclude by Theorem 2.1.47.

Based on the argument of Lemma 5.2.5, the required bounds are precisely described below equation (5.3.16) in the previous Proposition. This completes the proof. □

5.3.2 Counting invariants of (γ, η)-pairs

Let $\beta \in \mathrm{N}_1(\mathcal{X})$ be a class. First, we locate the walls for $(\gamma, \eta) \in \mathbf{R}_{>0} \times \mathbf{R}$ where the notion of $(\mathtt{T}_{\zeta_{\gamma,\eta}}, \mathtt{F}_{\zeta_{\gamma,\eta}})$-pair of class $(-1, \beta', c') \in \mathbf{Z} \oplus \mathrm{N}_{\leq 1}(\mathcal{X})$ with $\beta' \leq \beta$ could change.

Lemma 5.3.22. Let $\beta \in \mathrm{N}_1(\mathcal{X})$. The torsion pair $(\mathtt{T}_{\zeta_{\gamma,\eta}}, \mathtt{F}_{\zeta_{\gamma,\eta}})$ is constant on the connected components of $(\mathbf{R}_{>0} \times \mathbf{R}) \smallsetminus (\mathrm{V}_\beta \times \mathbf{R})$, which are finite in number, where

$$\mathrm{V}_\beta = \left\{ -\frac{\deg(\beta' \cdot \mathrm{A})}{|\operatorname{ch}_2(\Psi(\beta' \cdot \mathrm{A})) \cdot \omega|_{\mathrm{Y}}} : 0 < \beta' \leq \beta \right\} \cap \mathbf{R}_{>0}. \tag{5.3.19}$$

Proof. This follows from the argument of Lemma 5.2.7. Alternatively, a ζ-semistable subobject $0 \neq \mathrm{T} \hookrightarrow \mathrm{E}$ of a $\zeta_{\gamma,\eta}$-pair E of class $(-1, \beta, c)$ defines a γ-wall for β if and only if $\mathrm{L}_\gamma(\mathrm{T}) = 0$. Solving for $\gamma > 0$ and using $0 < \beta_{\mathrm{T}} \leq \beta$ yields the result. □

Recall that $\mathrm{W}_\beta = (1/d_\beta!)\mathbf{Z} \subset \mathbf{R}$ denotes the set of walls for $\delta \in \mathbf{R}$ where the notion of δ-pair of class $(-1, \beta', c')$ with $\beta' \leq \beta$ could change. Here $d_\beta = a_1(\beta) \in \mathbf{Z}_{>0}$.

Corollary 5.3.23. For each $\gamma \in \mathrm{V}_\beta$ the notion of $\zeta_{\gamma,\eta}$-pair of class $(-1, \beta, c)$ is locally constant on $\{\gamma\} \times \mathbf{R} \smallsetminus \mathrm{W}_\beta$.

Second, we define DT-type invariants virtually counting $\zeta_{\gamma,\eta}$-semistable sheaves and $(\mathtt{T}_{\zeta_{\gamma,\eta}}, \mathtt{F}_{\zeta_{\gamma,\eta}})$-pairs. As for δ-pairs, this is done by applying the integration map. We present these definitions for zero and non-zero rank separately.

Rank 0 Let $(a, b) \in \mathbf{R}^2$. Consider the subcategory $\mathcal{M}^{\mathrm{ss}}_\zeta(a, b) \subset \mathrm{Coh}_{\leq 1}(\mathcal{X})$ of ζ-semistable sheaves of slope (a, b). By Proposition 5.3.21, it defines an element $\mathbf{1}_{\mathrm{SS}^\zeta(a,b)}$ in the graded Hall algebra $\mathrm{H}_{\mathrm{gr}}(\mathtt{C})$ that is decompositionally finite. In particular, we obtain a graded-regular element

$$\eta_{\mathrm{SS}^\zeta(a,b)} = (\mathbf{L} - 1) \log \mathbf{1}_{\mathrm{SS}^\zeta(a,b)} \in \mathrm{H}_{\mathrm{gr,reg}}(\mathtt{C}) \tag{5.3.20}$$

by Theorem 4.3.11. Projecting to the semi-classical quotient $\mathrm{H}_{\mathrm{gr,sc}}(\mathtt{C})$, we define DT-type invariants $\mathrm{J}^\zeta_{(\beta,c)} \in \mathbf{Q}$ by the formula

$$\sum_{\zeta(\beta,c)=(a,b)} \mathrm{J}^\zeta_{(\beta,c)} z^\beta q^c := \mathrm{I}\left(\eta_{\mathrm{SS}^\zeta(a,b)}\right) \in \mathbf{Q}\{\mathrm{N}(\mathcal{X})\}. \tag{5.3.21}$$

These invariants 'count' ζ-semistable objects of slope (a,b).

Rank -1 Let $(\beta,c) \in \mathrm{N}_{\leq 1}(\mathcal{X})$, and let $(\gamma,\eta) \in \mathbf{R}_{>0} \times \mathbf{R}$ be away from any wall. By Lemmas 4.2.20 and 5.3.19, obtain an element $(\mathbf{L}-1)[\underline{\mathtt{P}}_{\zeta_{\gamma,\eta}}(\beta,c) \subset \underline{\mathtt{C}}] \in \mathrm{H}_{\mathrm{reg}}(\mathtt{C})$. Again, projecting to the semi-classical quotient $\mathrm{H}_{\mathrm{sc}}(\mathtt{C})$ and applying the integration morphism, we define integer DT-type invariants

$$\mathrm{DT}^{\zeta_{\gamma,\eta}}_{(\beta,c)} z^\beta q^c s := \mathrm{I}\big((\mathbf{L}-1)[\underline{\mathtt{P}}_{\zeta_{\gamma,\eta}}(\beta,c) \subset \underline{\mathtt{C}}]\big), \tag{5.3.22}$$

where $s = t^{-[\mathcal{O}_\mathcal{X}]}$ as before. These invariants 'count' $(\mathtt{T}_{\zeta_{\gamma,\eta}},\mathtt{F}_{\zeta_{\gamma,\eta}})$-pairs.

Remark 5.3.24. Note that the J^ζ-invariants do not depend on $(\gamma,\eta) \in \mathbf{R}_{>0}\times\mathbf{R}$, whereas the invariants $\mathrm{DT}^{\zeta_{\gamma,\eta}}$ *do* depend on these. This is incorporated in their notation.

Third, we define generating series for these invariants. These are motivated by the arguments in section 5.2.3. Let $\beta \in \mathrm{N}_1(\mathcal{X})$. We define the following series in $\mathbf{Q}\{\mathrm{N}(\mathcal{X})\}$:

$$\mathrm{DT}^{\zeta_{\gamma,\eta}}_{\leq\beta} := \sum_{\beta'\leq\beta} \sum_{c\in\mathrm{N}_0(\mathcal{X})} \mathrm{DT}^{\zeta_{\gamma,\eta}}_{(\beta',c)} z^{\beta'} q^c, \tag{5.3.23}$$

$$\mathrm{J}^\zeta(a,b)_{\leq\beta} := \sum_{\beta'\leq\beta} \sum_{\substack{c\in\mathrm{N}_0(\mathcal{X}) \\ \zeta_{\gamma,\eta}(\beta',c)=(a,b)}} \mathrm{J}^\zeta_{(\beta',c)} z^{\beta'} q^c. \tag{5.3.24}$$

These series are elements in smaller subrings of $\mathbf{Q}\{\mathrm{N}(\mathcal{X})\}$.

Lemma 5.3.25. We have $\mathrm{DT}^{\zeta_{\gamma,\eta}}_{\leq\beta} \in \mathbf{Z}[\mathrm{N}(\mathcal{X})]_{\mathrm{L}_\gamma}$ and $\mathrm{J}^\zeta(\gamma,\eta)_{\leq\beta} \in \mathbf{Q}[\mathrm{N}(\mathcal{X})]$.

Proof. The set $\{\beta' \in \mathrm{N}_1(\mathcal{X}) \mid \beta' \leq \beta\}$ is finite by Lemma 2.1.38. The first claim follows from Proposition 5.3.19. The second claim follows from part 2 of Proposition 5.3.21. $\square$

Fix a class $(\beta,c) \in \mathrm{N}_{\leq 1}(\mathcal{X})$. We now show that the invariant $\mathrm{DT}^{\zeta_{\gamma,\eta}}_{(\beta,c)}$ coincides with the stable pair invariant $\mathrm{PT}_\mathcal{X}(\beta,c)$ for $0<\gamma\ll 1$.

Lemma 5.3.26. Let $0<\epsilon\ll 1$. An object $\mathrm{E} \in \mathtt{A}$ of class $(-1,\beta,c)$ is a $(\mathtt{T}_{\zeta_{\epsilon,\eta}},\mathtt{F}_{\zeta_{\epsilon,\eta}})$-pair if and only if it is a $(\mathtt{T}_{\mathrm{PT}},\mathtt{F}_{\mathrm{PT}})$-pair. In particular,

$$\mathrm{DT}^{\zeta_{\epsilon,\eta}}_{(\beta,c)} = \mathrm{PT}_\mathcal{X}(\beta,c) \tag{5.3.25}$$

for all $\eta \in \mathbf{R}$ and for all such ϵ small enough.

Proof. Recall that $\mathtt{T}_{\mathrm{PT}} = \mathrm{Coh}_0(\mathcal{X})$, that $\mathtt{F}_{\mathrm{PT}} = \mathrm{Coh}_1(\mathcal{X})$, and that $(\mathtt{T}_{\mathrm{PT}},\mathtt{F}_{\mathrm{PT}})$-pairs are precisely stable pairs by Example 4.1.17. We have $\mathrm{Coh}_0(\mathcal{X}) \subset \mathtt{T}_{\zeta_{\gamma,\eta}}$ for all $(\gamma,\eta) \in \mathbf{R}_{>0}\times\mathbf{R}$. Conversely, if $0<\epsilon\ll 1$ is small enough, then $\zeta_{\epsilon,\eta}(\mathrm{T}) < (0,0)$ for any pure one-dimensional sheaf T such that $\beta_\mathrm{T} \leq \beta$, so $\mathrm{T} \notin \mathtt{T}_{\zeta_{\epsilon,\eta}}$. This completes the proof. $\square$

There is a unique curve class and a unique point class associated to each γ-wall, provided that the amples $(A, \omega) \in \mathrm{Amp}(X) \times \mathrm{Amp}(Y)$ are chosen sufficiently generally.

Lemma 5.3.27. If $A \in N^1(X)$ is general and $\omega \in N^1(Y)_{\mathbf{R}}$ is very general, then for each $\gamma \in V_\beta$ there is up to scaling a unique class $\beta_\gamma \in N_1(\mathcal{X})$ with $\beta_\gamma \leq \beta$ such that

$$L_\gamma(A \cdot \beta_\gamma) = 0. \tag{5.3.26}$$

The class $c_\gamma := \beta_\gamma \cdot A \in N_0(\mathcal{X})$ is up to scaling the unique class such that $L_\gamma(c_\gamma) = 0$.

Recall that $L_\gamma(c) = \deg(c) + \gamma|\operatorname{ch}_2(\Psi(c)) \cdot \omega|_Y \in \mathbf{R}$ where $c \in N_0(\mathcal{X})$.

Proof. If $L_\gamma(A \cdot \beta') = 0$, then $\beta' \notin N_{\mathrm{mr}}(\mathcal{X})$ since multi-regular classes have $L_\gamma(A \cdot \beta') = -1$. Thus we restrict our search to $S = \{\beta' \mid 0 < \beta' \leq \beta, \beta' \notin N_{\mathrm{mr}}(\mathcal{X})\}$. For $\beta' \in S$, define

$$\gamma_{\beta'}(A, \omega) = -\frac{\deg(\beta' \cdot A)}{|\operatorname{ch}_1(\Psi(\beta')) \cdot A \cdot \omega|_Y} \in \mathbf{Q}_{<0}. \tag{5.3.27}$$

The denominator is non-zero because $\beta' \notin N_{\mathrm{mr}}(\mathcal{X})$. Moreover, note that

$$\operatorname{ch}_1(\Psi(\beta')) \cdot A = \operatorname{ch}_2(\Psi(\beta') \cdot A) = \operatorname{ch}_2(\Psi(\beta' \cdot A)) \tag{5.3.28}$$

by the Kleiman-trick of Lemma 2.1.37 and the fact that tensoring by the pullback of a line bundle from X commutes with the McKay equivalence. Thus, per construction, $\gamma_{\beta'}(A, \omega)$ is the unique number for which $L_{\gamma_{\beta'}}(\beta' \cdot A) = 0$ given (A, ω).

Take $\beta', \beta'' \in S$ and assume that β' is not proportional to β''. The locus of $(A, \omega) \in N^1(X) \times N^1(Y)$ for which $\gamma_{\beta'}(A, \omega) \neq \gamma_{\beta''}(A, \omega)$ is an open algebraic subset of $N^1(X) \times N^1(Y)$. It is non-empty because there exist A for which $\gamma_{\beta'}(A, \omega) = 0 \neq \gamma_{\beta''}(A, \omega)$. Since S is a finite set, taking (A, ω) to be ample classes in the intersection of these finitely many open subsets we have the first claim.

Let $c \in N_0(\mathcal{X})$. For the second claim, recall that $\gamma > 0$. It follows that $L_\gamma(c) = 0$ implies that $c \notin \Phi(N_0(Y)) \cup \deg^{-1}(0)$. Thus we restrict our search to elements of the countable set $T = \{c \in N_0(\mathcal{X}) \mid \Psi(c) \notin N_0(Y) \text{ and } \deg(c) \neq 0\}$. Define

$$\gamma_c(\omega) = -\frac{\deg(c)}{|\operatorname{ch}_2(\Psi(c)) \cdot \omega|_Y}, \tag{5.3.29}$$

so that $\gamma_c(\omega)$ is the unique number such that $L_{\gamma_c(\omega)}(c) = 0$. If $c, c' \in N_0(\mathcal{X})$ are not proportional, then neither are $\operatorname{ch}_2(\Psi(c))$ and $\operatorname{ch}_2(\Psi(c'))$, and so the condition $\gamma_c(\omega) \neq \gamma_{c'}(\omega)$ defines a non-empty Zariski open subset of $N^1(Y)_{\mathbf{R}}$. Taking ω to lie in the complement of the countably many such condition gives the claim. ☐

Lemma 5.3.28. Let $\gamma \in \mathrm{V}_\beta$, and let S be the set of ζ-semistable, pure one-dimensional sheaves F with $\beta_{\mathrm{F}} = d\beta_\gamma \leq \beta$. Then $c(\mathrm{S}) = \{c_{\mathrm{F}} \in \mathrm{N}_0(\mathcal{X}) \mid \mathrm{F} \in \mathrm{S}\}$ is weakly L_γ-bounded.

Proof. Because $\mathrm{L}_\gamma(\beta_{\mathrm{F}} \cdot \mathrm{A}) = d\mathrm{L}_\gamma(\beta_\gamma \cdot \mathrm{A}) = 0$ we find $\zeta_{\gamma,\eta}(\mathrm{F}) = (0, \nu(\mathrm{F}) + \eta)$ and so $\mathrm{F} \in \mathtt{F}_{\zeta_{\gamma,\infty}}$. Note that $\zeta_{\gamma,\eta}(\mathrm{F} \otimes \mathrm{A}^{\otimes k}) = (0, \nu(\mathrm{F} \otimes \mathrm{A}^{\otimes k}) + \eta)$. Recall that $\nu(\mathrm{F} \otimes \mathrm{A}^{\otimes k}) = \nu(\mathrm{F}) + k$ for all one-dimensional F and $k \in \mathbf{Z}$. Thus, for $n \gg 0$ we have $\mathrm{F} \otimes \mathrm{A}^{\otimes n} \in \mathtt{T}_0 \cap \mathtt{F}_{\zeta_{\gamma,\infty}}$. The claim then follows by Lemma 5.3.18. □

5.3.3 Crossing the γ-wall

Let $\beta \in \mathrm{N}_1(\mathcal{X})$ be a curve class. We analyse what happens to the generating series $\mathrm{DT}^{\zeta_{\gamma,\eta}}_{\leq\beta} \in \mathbf{Z}[\mathrm{N}(\mathcal{X})]_{\mathrm{L}_\gamma}$ of $(\mathtt{T}_{\zeta_{\gamma,\eta}}, \mathtt{F}_{\zeta_{\gamma,\eta}})$-pair invariants when γ crosses a wall in V_β.

Unfortunately, the γ-walls are too large for the wall-crossing formula to apply directly. Instead, we vary η to $\pm\infty$ and consider the associated η-wall crossings, each of which are 'small' enough for our framework to apply. It turns out that the limit $\eta \to -\infty$ yields the generating series with $\gamma - \epsilon$ whereas $\eta \to \infty$ yields the series with $\gamma + \epsilon$ for some $0 < \epsilon \ll 1$. Indirectly, this realises the γ-wall crossing.

First, we collect further finiteness results allowing us to apply the wall-crossing formula and establish an upper bound on the degree of $\zeta_{\gamma,\eta}$-pairs of a fixes curve class.

Lemma 5.3.29. Let $\beta \in \mathrm{N}_1(\mathcal{X})$ be a curve class, let $\gamma \in \mathrm{V}_\beta$ be a wall, and let $c_0 \in \mathrm{N}_0(\mathcal{X})$.

1. There exist $m_{(\beta,c_0),\gamma}$ and $\mathrm{M}_{(\beta,c_0),\gamma}$ in $\mathbf{Z}$ such that if there exists a decomposition
$$(\beta, c_0 + kc_\gamma) = (\beta', c') + (d\beta_\gamma, c) \tag{5.3.30}$$
where $k \in \mathbf{Z}$, $d \in \mathbf{Z}_{>0}$, $\beta' \in \mathrm{N}_1(\mathcal{X})$ and $c, c' \in \mathrm{N}_0(\mathcal{X})$, such that
$$\underline{\mathtt{P}}_{\zeta_{\gamma,\eta}}(\beta', c') \neq \varnothing \neq \underline{\mathcal{M}}^{\mathrm{ss}}_\zeta(d\beta_\gamma, c), \tag{5.3.31}$$
then the degree of c' is bounded $m_{(\beta,c_0),\gamma} \leq \deg(\beta', c') \leq \mathrm{M}_{(\beta,c_0),\gamma}$.
2. There exists an increasing linear function $\mathrm{A}^-_{(\beta,c_0),\gamma}\colon \mathbf{R}_{>0} \to \mathbf{R}$ such that if
$$\deg(\beta, c_0 + kc_\gamma) \leq \mathrm{A}^-_{(\beta,c_0),\gamma}(\eta) \tag{5.3.32}$$
the notion of $\zeta_{\gamma,\eta'}$-pair of class $(\beta, c_0 + kc_\gamma)$ is unchanged for any $\eta' \geq \eta \geq 0$:
$$\underline{\mathtt{P}}_{\zeta_{\gamma,\eta}}(\beta, c_0 + kc_\gamma) = \underline{\mathtt{P}}_{\zeta_{\gamma,\eta'}}(\beta, c_0 + kc_\gamma). \tag{5.3.33}$$
3. There exists an increasing linear function $\mathrm{A}^+_{(\beta,c_0),\gamma}\colon \mathbf{R}_{<0} \to \mathbf{R}$ such that if
$$\mathrm{A}^+_{(\beta,c_0),\gamma}(\eta) \leq \deg(\beta, c_0 + kc_\gamma) \tag{5.3.34}$$

the notion of $\zeta_{\gamma,\eta'}$-pair of class $(\beta, c_0 + kc_\gamma)$ is unchanged for any $\eta' \leq \eta \leq 0$:

$$\underline{\mathrm{P}}_{\zeta_{\gamma,\eta}}(\beta, c_0 + kc_\gamma) = \underline{\mathrm{P}}_{\zeta_{\gamma,\eta'}}(\beta, c_0 + kc_\gamma). \tag{5.3.35}$$

4. There exist constants $\mathrm{C}^-_{(\beta,c_0),\gamma}$ and $\mathrm{C}^+_{(\beta,c_0),\gamma}$ in $\mathbf{Z}$ such that

$$\deg(\beta, c_0 + kc_\gamma) \leq \mathrm{C}^-_{(\beta,c_0),\gamma} \Longrightarrow \mathrm{P}_{\zeta_{\gamma,\eta}}(\beta, c_0 + kc_\gamma) = \varnothing \text{ for } \eta \geq 0 \tag{5.3.36}$$

$$\deg(\beta, c_0 + kc_\gamma) \geq \mathrm{C}^+_{(\beta,c_0),\gamma} \Longrightarrow \mathrm{P}_{\zeta_{\gamma,\eta}}(\beta, c_0 + kc_\gamma) = \varnothing \text{ for } \eta \leq 0 \tag{5.3.37}$$

Proof. For the first claim, note that c lies in a weakly L_γ-bounded set by Lemma 5.3.28. Thus there exists a constant C_β such that $\mathrm{L}_\gamma(c) \geq \mathrm{C}_\beta$. Since $\mathrm{L}_\gamma(c_\gamma) = 0$ we find

$$\mathrm{L}_\gamma(c') \leq \mathrm{L}_\gamma(c_0 + kc_\gamma) - \mathrm{C}_\beta = \mathrm{L}_\gamma(c_0) - \mathrm{C}_\beta. \tag{5.3.38}$$

By Lemma 5.3.19, it follows that there is a finite set of possible values for $c' \in \mathrm{N}_0(\mathcal{X})$. The bounds are the minimum and maximum of $\deg(\beta', c')$ as c' runs through this set.

For the second and third claim, observe that $\underline{\mathrm{P}}_{\zeta_{\gamma,\eta}}(\beta, c_0 + kc_\gamma)$ can change at the wall (γ, η) only if there exists a decomposition

$$(\beta, c_0 + kc_\gamma) = (\beta', c') + (d\beta_\gamma, c) \tag{5.3.39}$$

with

$$\underline{\mathrm{P}}_{\zeta_{\gamma,\eta}}(\beta', c') \neq \varnothing \neq \underline{\mathcal{M}}^{\mathrm{ss}}_\zeta(d\beta_\gamma, c) \tag{5.3.40}$$

and such that $\nu(d\beta_\gamma, c) = \eta$. Recall that $\nu(\beta, c) \equiv \deg(\beta, c)/d_\beta$, where $d_\beta = \deg(\beta \cdot \mathrm{A})$ by Lemma 2.1.37, and recall that $1 \leq d\deg(\beta_\gamma \cdot \mathrm{A}) \leq d_\beta$ since $d\beta_\gamma \leq \beta$. From the first part, and the relation $\deg(c_0 + kc_\gamma) = \deg(\beta', c') + \deg(\beta, c)$, we deduce the following:

1. Assume $\eta \geq 0$. We find that

$$\begin{aligned} \deg(c_0 + kc_\gamma) &\geq m_{(\beta,c_0),\gamma} + d\deg(\mathrm{A} \cdot \beta_\gamma)\eta \\ &\geq m_{(\beta,c_0),\gamma} + \eta. \end{aligned} \tag{5.3.41}$$

   Thus, the increasing function $\mathrm{A}^-_{(\beta,c_0),\gamma}(\eta) = m_{(\beta,c_0),\gamma} + \eta - 1$ yields the claim.

2. Assume $\eta \leq 0$. We find that

$$\begin{aligned} \deg(c_0 + kc_\gamma) &\leq \mathrm{M}_{(\beta,c_0),\gamma} + d\deg(\mathrm{A} \cdot \beta_\gamma)\eta \\ &\leq \mathrm{M}_{(\beta,c_0),\gamma} + \eta. \end{aligned} \tag{5.3.42}$$

   Thus, the increasing function $\mathrm{A}^+_{(\beta,c_0),\gamma}(\eta) = \mathrm{M}_{(\beta,c_0),\gamma} + \eta + 1$ yields the claim.

This proves the second and third claims.

For the fourth claim, we have $\underline{\mathrm{P}}_{\zeta_{\gamma,\eta}}(\beta, c_0 + kc_\gamma) = \underline{\mathrm{P}}_{\zeta_{\gamma,0}}(\beta, c_0 + kc_\gamma)$ for all $\eta \geq 0$ provided that we choose the constant $\mathrm{C}^-_{(\beta,c_0),\gamma}$ to be at least $\mathrm{A}^-_{(\beta,c_0),\gamma}(0)$. This follows by the second part of this lemma. Similarly, by the third part of this lemma, we have $\underline{\mathrm{P}}_{\zeta_{\gamma,\eta}}(\beta, c_0 + kc_\gamma) = \underline{\mathrm{P}}_{\zeta_{\gamma,0}}(\beta, c_0 + kc_\gamma)$ for all $\eta \leq 0$ provided that we choose the constant $\mathrm{C}^+_{(\beta,c_0),\gamma}$ to be at most $\mathrm{A}^+_{(\beta,c_0),\gamma}(0)$. Moreover, by Lemma 5.3.19, the set

$$\mathrm{S} = \{c_0 + kc_\gamma \in \mathrm{N}_0(\mathcal{X}) \mid k \in \mathbf{Z}, \underline{\mathrm{P}}_{\zeta_{\gamma,0}}(\beta, c_0 + kc_\gamma) \neq \varnothing\} \tag{5.3.43}$$

is L_γ-bounded, i.e., bounding L_γ from above determines a finite subset of S. But $\mathrm{L}_\gamma(c_0 + kc_\gamma) = \mathrm{L}_\gamma(c_0)$ is the same for each element of S. Thus S is finite.

Let $m = \min\{\deg(s) \mid s \in \mathrm{S}\}$ and $\mathrm{M} = \max\{\deg(s) \mid s \in \mathrm{S}\}$ denote the minimum and maximum degrees of elements in S respectively. The constants

$$\begin{aligned} \mathrm{C}^-_{(\beta,c_0),\gamma} &:= \mathrm{A}^-_{(\beta,c_0),\gamma}(0) + m - 1 \\ \mathrm{C}^+_{(\beta,c_0),\gamma} &:= \mathrm{A}^+_{(\beta,c_0),\gamma}(0) + \mathrm{M} + 1 \end{aligned} \tag{5.3.44}$$

satisfy the property in the claim. This completes the proof. ☐

Remark 5.3.30. For classes $c_0 + kc_\gamma \in \mathrm{N}_0(\mathcal{X})$ such that $\underline{\mathrm{P}}_{\zeta_{\gamma,0}}(\beta, c_0 + kc_\gamma) \neq 0$, we find the constraints

$$\mathrm{C}^-_{(\beta,c_0),\gamma} < \deg(c_0 + kc_\gamma) < \mathrm{C}^+_{(\beta,c_0),\gamma}. \tag{5.3.45}$$

As a sanity check, note that $\mathrm{C}^+_{(\beta,c_0),\gamma)} - \mathrm{C}^-_{(\beta,c_0),\gamma)} \geq 2$, so such classes can indeed exist.

Let $\beta \in \mathrm{N}_1(\mathcal{X})$ be a curve class. Recall that if $\gamma \notin \mathrm{V}_\beta$ is not a wall for β, then the notion of $\zeta_{\gamma,\eta}$-pair is independent of η. In that we case, we may simply write $\mathrm{DT}^{\zeta_\gamma}_{(\beta,c)}$ instead of $\mathrm{DT}^{\zeta_{\gamma,\eta}}_{(\beta,c)}$. However, if γ is a wall for β, the notion of $\zeta_{\gamma,\eta}$-pair *does* depend on η. The following result shows that letting $\eta \to \pm\infty$ lands us on either side of the wall γ.

Proposition 5.3.31. Let $\beta \in \mathrm{N}_1(\mathcal{X})$ be a curve class, let $\gamma \in \mathrm{V}_\beta$ be a wall for β, and let $0 < \epsilon \ll 1$ be such that $[\gamma_-, \gamma_+] \cap \mathrm{V}_\beta = \{\gamma\}$ where $\gamma_\pm = \gamma \pm \epsilon$. Then we have

$$\mathrm{DT}^{\zeta_{\gamma_\pm}}_{(\beta,c_0+kc_\gamma)} = \mathrm{DT}^{\zeta_{\gamma,\pm\infty}}_{(\beta,c_0+kc_\gamma)} \tag{5.3.46}$$

for all $c_0 \in \mathrm{N}_0(\mathcal{X})$ and all $k \in \mathbf{Z}$.

To ease notation, we write the ζ-slope as $\zeta_{\gamma,\eta}(\mathrm{T}) = (-1 - \gamma f(\mathrm{T}), \nu(\mathrm{T}) + \eta)$.

Proof. Let $c_0 \in \mathrm{N}_0(\mathcal{X})$. The result follows if we show the stronger assertion that the categories of pairs $\mathtt{P}_{\zeta_{\gamma_+}}(\beta, c_0 + kc_\gamma) = \mathtt{P}_{\zeta_{\gamma,\eta}}(\beta, c_0 + kc_\gamma)$ are equal. By parts 2 and 3 of

Lemma 5.3.29, it suffices to show

$$\mathtt{P}_{\zeta_{\gamma_+,\eta'}}(\beta, c_0 + kc_\gamma) = \mathtt{P}_{\zeta_{\gamma,\eta}}(\beta, c_0 + kc_\gamma) \tag{5.3.47}$$

are equal for $\eta \gg 0$, and similarly with γ_- in place of γ_+ and $\eta \ll 0$ in place of $\eta \gg 0$. Since $\gamma_\pm \notin \mathrm{V}_\beta$, the left hand category is independent of η'. Thus we may take $\eta' := \eta$.

We only prove the statement for γ_+ as the other case is similar.

Let E be a $\zeta_{\gamma,\eta}$-pair of class $(\beta, c_0 + kc_\gamma)$. It is not hard to see that $\mathtt{T}_{\zeta_{\gamma',\eta}} \subseteq \mathtt{T}_{\zeta_{\gamma'',\eta}}$ if $\gamma' \le \gamma''$, and hence $\mathtt{F}_{\zeta_{\gamma',\eta}} \supseteq \mathtt{F}_{\zeta_{\gamma'',\eta}}$ by Hom-vanishing. It follows that $\mathrm{Hom}(\mathrm{E}, \mathtt{F}_{\zeta_{\gamma_+,\eta}}) = 0$.

We claim that $\mathrm{Hom}(\mathtt{T}_{\zeta_{\gamma_+,\eta}}, \mathrm{E}) = 0$ as well and, hence, that E is a $\zeta_{\gamma_+,\eta}$-pair too. Suppose to the contrary that there exists a non-zero injection $\mathrm{T} \hookrightarrow \mathrm{E}$ with $\mathrm{T} \in \mathtt{T}_{\zeta_{\gamma_+,\eta}}$, i.e., $\zeta_{\gamma_+,\eta}(\mathrm{T}) \ge (0,0)$ and T destabilizes E as $\zeta_{\gamma_+,\eta}$-pair. Without loss of generality, we may assume that T is ζ-semistable and pure of dimension one. Since $\gamma_+ \notin \mathrm{V}_\beta$, the inequality $\zeta_{\gamma_+,\eta}(\mathrm{T}) \ge (0,0)$ reads $0 < -1 - (\gamma + \epsilon) f(\mathrm{T})$.

Note that E is a $\zeta_{\gamma,\eta}$-pair, so we also have $\zeta_{\gamma,\eta}(\mathrm{T}) < (0,0)$. Because $\eta \gg 0$, this implies that $-1 - \gamma f(\mathrm{T}) < 0$. Both bounds together yield $\gamma + \epsilon > -f(\mathrm{T})^{-1} > \gamma$. By Lemma 5.3.22 this means that T induces a γ-wall for β, which is a contradiction.

The proof that $\mathtt{P}_{\zeta_{\gamma_+,\eta'}}(\beta, c_0 + kc_\gamma) \subseteq \mathtt{P}_{\zeta_{\gamma,\eta}}(\beta, c_0 + kc_\gamma)$ follows by a dual argument. □

The following is the key result in order to realise the γ-wall crossing.

Lemma 5.3.32. Let $\beta \in \mathrm{N}_1(\mathcal{X})$ be a curve class, let $c_0 \in \mathrm{N}_0(\mathcal{X})$, and let $\gamma \in \mathrm{V}_\beta$ be a wall for β. There exists an increasing linear function $\mathrm{B}_{(\beta,c_0),\gamma}(\eta)$ such that

$$b(k) := \mathrm{DT}^{\zeta_{\gamma,\eta}}_{(\beta,c_0+kc_\gamma)} - \mathrm{DT}^{\zeta_{\gamma,-\infty}}_{(\beta,c_0+kc_\gamma)} \tag{5.3.48}$$

is a quasi-polynomial in k provided $k \le \mathrm{B}_{(\beta,c_0),\gamma}(\eta)$ and $\eta \ge d_\beta$.

We illustrate the above bound on quasi-polynomial behaviour in a simple example. This also motivates the *key claim* made in the proof below. Let $p(k)$ be a quasi-polynomial in a single variable $k \in \mathbf{Z}$, and consider the generating function

$$f(q) = \sum_{k=k_0}^{-\infty} p(k) q^k. \tag{5.3.49}$$

In order to claim that the coefficient of q^k in $f(q)$ is a quasi-polynomial when $k \le \mathrm{B}$, it suffices to know that $\mathrm{B} \le k_0$. In other words, if B is a lower bound on k_0, it is also a lower bound on the set of (degrees of) powers of q for which the coefficient function *fails* to be a quasi-polynomial.

In the proof, we find the function $\mathrm{B}_{(\beta,c_0),\gamma}(\eta)$ by establishing such a bound $\mathrm{B} \le k_0$.

Proof. Since $\gamma \in \mathrm{V}_\beta$, the walls for $\zeta_{\gamma,\eta}$-pairs of class $(-1, \beta', c)$ with $\beta' \leq \beta$ are given by the η-walls W_β for Nironi stability by Corollary 5.3.23. By Proposition 5.2.18, we have a wall-crossing formula for W_β. Iterating this wall-crossing formula yields

$$\mathrm{DT}^{\zeta_{\gamma},-\infty}_{\leq\beta} s = \prod_{\eta' \in \mathrm{W}_\beta \cap (-\infty,\eta)} \exp(\{-\mathrm{J}^{\zeta}_{\beta_\gamma}(\eta'), -\})\mathrm{DT}^{\zeta_{\gamma,\eta}}_{\leq\beta} s \tag{5.3.50}$$

where the product is taken in *decreasing* order of η'; this explains the minus sign in the exponent. Furthermore, we have defined

$$\mathrm{J}^{\zeta}_{\beta_\gamma}(\eta') = \sum_{\substack{d \in \mathbf{Z}_{\geq 1}, c \in \mathrm{N}_0(\mathcal{X}) \\ d\beta_\gamma \leq \beta \\ \nu(d\beta_\gamma, c) = \eta'}} \mathrm{J}^{\zeta}_{(d\beta_\gamma, c)} z^{d\beta_\gamma} q^c. \tag{5.3.51}$$

To prove this result, we follow the strategy of the proof of Theorem 5.1.3.

Expanding the exponential, and substituting the expressions from equation (5.3.23), we collect all terms contributing to the coefficient of $z^\beta q^{c_0 + kc_\gamma} s$ on the right hand side. The terms of this sum are described as follows. Fix an $r \geq 1$, a sequence $(d_i)_{i=1}^r$ in $\mathbf{Z}_{\geq 1}$, a sequence $(c_i)_{i=1}^r$ in $\mathrm{N}_0(\mathcal{X})$, and a class $\alpha' = (\beta', c') \in \mathrm{N}_{\leq 1}(\mathcal{X})$, satisfying

- $\beta = \beta' + \sum_{i=1}^r d_i \beta_\gamma$,
- $c' + \sum_{i=1}^r c_i \equiv c_0 \pmod{c_\gamma}$,
- $\nu(d_r\beta_\gamma, c_r) \leq \cdots \leq \nu(d_1\beta_\gamma, c_1) < \eta$,
- $\mathrm{J}^{\zeta}_{(d_i\beta_\gamma, c_i)} \neq 0$ for all $1 \leq i \leq r$,
- $\mathrm{DT}^{\zeta_{\gamma,\eta}}_{\alpha'} \neq 0$.

The non-zero term in the coefficients of $z^\beta q^{c_0 + kc_\gamma} s$ associated with this data is

$$\begin{aligned}\mathrm{T}(r, (d_i), (c_i), \alpha') z^\beta q^{c' + \sum c_i} s = \mathrm{A}_{(d_i),(c_i)} &\{-\mathrm{J}^{\zeta}_{(d_r\beta_\gamma, c_r)} z^{d_r\beta_\gamma} q^{c_r}, -\} \circ \\ &\cdots \circ \{-\mathrm{J}^{\zeta}_{(d_1\beta_\gamma, c_1)} z^{d_1\beta_\gamma} q^{c_1}, -\}(\mathrm{DT}^{\zeta_{\gamma,\eta}}_{(\beta', c')} z^{\beta'} q^{c'} s)\end{aligned}$$

where $\mathrm{A}_{(d_i),(c_i)}$ is a factor arising from the exponential:

$$\mathrm{A}_{(d_i),(c_i)} = \prod_{\nu \in \mathrm{W}_\beta} \frac{1}{|\{i \mid \nu(d_i\beta_\gamma, c_i) = \nu\}|!}. \tag{5.3.52}$$

Putting all these terms together, we have

$$\sum_{k\in\mathbf{Z}}\left(\mathrm{DT}^{\zeta_\gamma,\eta}_{(\beta,c_0+kc_\gamma)}-\mathrm{DT}^{\zeta_\gamma,-\infty}_{(\beta,c_0+kc_\gamma)}\right)q^{c_0+kc_\gamma}=\sum_{r,(d_i),(c_i),\alpha'}\mathrm{T}(r,(d_i),(c_i),\alpha')q^{c'+\sum c_i}. \tag{5.3.53}$$

We now analyse the T-terms. Expanding the Poisson brackets yields

$$\mathrm{T}(r,(d_i),(c_i),\alpha')=\mathrm{A}_{(d_i),(c_i)}\mathrm{B}_{(d_i),(c_i),\alpha'}\prod_{i=1}^{r}\mathrm{J}^{\zeta}_{(d_i\beta_\gamma,c_i)}\mathrm{DT}^{\zeta_\gamma,\eta}_{\alpha'}$$

where, letting $\alpha_i=(d_i\beta_\gamma,c_i)$ and $\alpha'=(\beta',c')$, we have

$$\mathrm{B}_{(d_i),(c_i),\alpha'}=(-1)^r\sigma^{\sum_{i<j}\chi(\alpha_j,\alpha_i)+\sum_i\chi(\alpha_i,\alpha'-[\mathcal{O}_{\mathcal{X}}])}\prod_{i=1}^{r}\chi(\alpha_i,-[\mathcal{O}_{\mathcal{X}}]+\alpha'+\sum_{j=1}^{i-1}\alpha_j).$$

The sign $(-1)^r$ arises because the wall-crossing is in decreasing order of slope.

We partition these T-terms in groups as follows. A *group* consists of the data of a class $\alpha'=(\beta',c')\in \mathrm{N}_{\leq 1}(\mathcal{X})$ and a sequence of positive integers $(d_i)_{i=1}^r$ satisfying the same conditions as above, a sequence $(\kappa_i)_{i=1}^r\in \mathrm{N}_0(\mathcal{X})/\mathbf{Z}(d_i\beta_\gamma\cdot \mathrm{A})$, and a subset $\mathrm{E}\subseteq\{1,\dots,r-1\}$. The κ_i are required to satisfy

$$\mathrm{J}^{\zeta}_{(d_i\beta_\gamma,c_i)}\neq 0 \text{ for } c_i\in\alpha_i \text{ and all } i=1,2,\dots,r. \tag{5.3.54}$$

Tensoring by the line bundle A induces an isomorphism $\underline{\mathcal{M}}^{\mathrm{ss}}_\zeta(\gamma,d)\cong\underline{\mathcal{M}}^{\mathrm{ss}}_\zeta(\gamma,d+\gamma\cdot\mathrm{A})$. It follows that the invariant $\mathrm{J}^{\zeta}_{(d_i\beta_\gamma,c_i)}$ is independent of the choice of representative $c_i\in\kappa_i$, as is its non-vanishing property. Thus we may write $\mathrm{J}^{\zeta}_{(d_i\beta_\gamma,\kappa_i)}:=\mathrm{J}^{\zeta}_{(d_i\beta_\gamma,c_i)}$.

Collecting all terms belonging to the group $(\alpha',(d_i),(\kappa_i),\mathrm{E})$, we obtain

$$\mathrm{C}(\alpha',(d_i),(\kappa_i),\mathrm{E})=\sum_{(c_i)}\mathrm{T}(r,(d_i),(c_i),\alpha')q^{c'+\sum c_i}, \tag{5.3.55}$$

where the sum is over all $c_i\in\mathrm{N}_0(\mathcal{X})$ such that

$$c_i\in\kappa_i, \tag{5.3.56}$$

$$\nu(d_r\beta_\gamma,c_r)\leq\cdots\leq\nu(d_2\beta_\gamma,c_2)\leq\nu(d_1\beta_\gamma,c_1)<\eta, \tag{5.3.57}$$

$$\nu(d_i\beta_\gamma,c_i)=\nu(d_{i+1}\beta_\gamma,c_{i+1})\Leftrightarrow i\in\mathrm{E}. \tag{5.3.58}$$

Note that for such a choice of c_i, the factor $\mathrm{A}_{(d_i),(c_i)}$ defined above depends only on E.

Indeed, set $\{n_i\} = \{1, \dots, r\} \setminus \mathrm{E}$ with $n_1 < \cdots < n_{r-|\mathrm{E}|}$. Then

$$\mathrm{A}_{\mathrm{E}} := \prod \frac{1}{(n_i - n_{i-1})!} = \mathrm{A}_{(d_i),(c_i)}. \tag{5.3.59}$$

We find that the contribution of the group $(\alpha', (d_i), (\kappa_i), \mathrm{E})$ is

$$\mathrm{C}(\alpha', (d_i), (\kappa_i), \mathrm{E}) = \mathrm{A}_{\mathrm{E}} \prod_{i=1}^{r} \mathrm{J}^{\zeta}_{(d_i\beta_\gamma, \kappa_i)} \mathrm{DT}^{\zeta_\gamma, \eta}_{\alpha'} \left(\sum_{c_i} \mathrm{B}_{(d_i),(c_i),\alpha'} q^{c' + \sum c_i} \right) \tag{5.3.60}$$

where the sum runs over all $c_i \in \mathrm{N}_0(\mathcal{X})$ as above. Now, for every choice of (d_i), (κ_i), and E, there exists a sequence (c_i^0) with $c_i^0 \in \kappa_i$ which is *maximal* in the sense that replacing any c_i^0 with $c_i^0 + d_i c_\gamma$ would violate one of (5.3.57) and (5.3.58).

We find

$$\mathrm{C}(\alpha', (d_i), (\kappa_i), \mathrm{E}) = \mathrm{A}_{\mathrm{E}} \prod_{i=1}^{r} \mathrm{J}^{\zeta}_{(d_i\beta_\gamma, \kappa_i)} \mathrm{DT}^{\zeta_\gamma, \eta}_{\alpha'} \left(\sum_{a_i} \mathrm{B}_{(d_i),(c_i^0 - a_i d_i c_\gamma),\alpha'} q^{c' + \sum c_i^0 - a_i d_i c_\gamma} \right)$$

where the sum is over the set $\mathrm{S}_{\mathrm{E}} = \{0 \le a_1 \le a_2 \le \ldots \le a_r \mid a_i \in \mathbf{Z},\ a_i = a_{i+1} \Leftrightarrow i \in \mathrm{E}\}$. Note that the coefficients of this expression depend quasi-polynomially on the a_i.

We want to find a bound B such that $\deg(c_0 + kc_\gamma) \le \mathrm{B}$ guarantees that the coefficient of $q^{c_0 + kc_\gamma}$ is a quasi-polynomial in k. As discussed above, it suffices to let B be a lower bound for the degrees of the exponents $c' + \sum c_i$ appearing in the right hand side of equation (5.3.53). In other words, we need

$$\mathrm{B} \le \max\left\{ \deg(\beta, c' + \sum c_i^0 - a_i d_i c_\gamma) \mid (a_i)_i \in \mathrm{S}_{\mathrm{E}} \right\}. \tag{5.3.61}$$

Since $a_i \ge 0$, $d_i \ge 1$, and $\deg(\beta_\gamma, c_\gamma) \ge 1$ it suffices to let B be a lower bound for $\deg(\beta, c' + \sum c_i^0)$. We proceed as follows. On the one hand, the maximality property of the c_i^0 is equivalent to the bound

$$\nu(d_i\beta_\gamma, c_i^0) \ge \eta - i, \tag{5.3.62}$$

for all $i = 1, 2, \dots, r$. Assuming, without loss of generality, that $\eta \ge d_\beta$, we find

$$\deg(d_i\beta_\gamma, c_i^0) \ge \nu(d_i\beta_\gamma, c_i^0) \ge \eta - d_\beta, \tag{5.3.63}$$

because $d_i d_{\beta_\gamma} \in \mathbf{Z}_{\ge 1}$ and $1 \le i \le r \le d_\beta$. In particular, we obtain the bound

$$\sum_{i=1}^{r} \deg(d_i\beta_\gamma, c_i^0) \ge \eta - d_\beta > 0. \tag{5.3.64}$$

On the other hand, since (β', c') is the class of a $\zeta_{\gamma,\eta}$-pair occurring as part of a $\zeta_{\gamma,\eta}$-pair of class $(\beta, c_0 + kc_\gamma)$, we have the bound $m_{(\beta,c_0),\gamma} \leq \deg(\beta', c')$ by the first part of Lemma 5.3.29. Together, we have the bound

$$m_{(\beta,c_0),\gamma} + \eta - d_\beta \leq \deg(\beta, c' + \sum_{i=1}^{r} c_i^0). \tag{5.3.65}$$

The *key claim* is that $b(k)$ is quasi-polynomial in k when the inequality

$$\deg(\beta, c_0 + kc_\gamma) \leq m_{(\beta,c_0),\gamma} + \eta - d_\beta \tag{5.3.66}$$

holds. For this to be true, note that there are only finitely many groups which contribute to equation (5.3.53), i.e., there are only finitely many non-trivial choices for the data $(r, \alpha', (d_i), (\kappa_i), \mathrm{E})$ where $\alpha' = (\beta', c')$. For the choice of $r, (d_i), \beta'$ and E, this is clear. For c' and (α_i), this follows from Lemma 5.3.19 and Proposition 5.3.28 respectively. Moreover, note that the bound in equation (5.3.66) is independent of this data.

In conclusion, the increasing linear function in η,

$$\mathrm{B}_{(\beta,c_0),\gamma}(\eta) = \frac{\eta + m_{(\beta,c_0),\gamma} - d_\beta - \deg(\beta, c_0)}{\deg(\beta_\gamma, c_\gamma)}, \tag{5.3.67}$$

satisfies the property that $b(k)$ in equation (5.3.48) is quasi-polynomial provided that $k \leq \mathrm{B}_{(\beta,c_0),\gamma}(\eta)$ and $\eta \geq d_\beta$. The function $\mathrm{B}_{(\beta,c_0),\gamma}(\eta)$ is well-defined because $\deg(\beta_\gamma, c_\gamma) > 0$ since $c_\gamma = \beta_\gamma \cdot \mathrm{A}$ and β_γ is effective. This completes the proof. □

Combining Lemmas 5.3.29 and 5.3.32, and letting η go to ∞, we obtain

Corollary 5.3.33. Let $\beta \in \mathrm{N}_1(\mathcal{X})$ be a curve class, let $c_0 \in \mathrm{N}_0(\mathcal{X})$, and let $\gamma \in \mathrm{V}_\beta$ be a wall for β. The function

$$\mathrm{F}(k) = \mathrm{DT}^{\zeta_\gamma,\infty}_{(\beta,c_0+kc_\gamma)} - \mathrm{DT}^{\zeta_\gamma,-\infty}_{(\beta,c_0+kc_\gamma)} \tag{5.3.68}$$

is a quasi-polynomial in $k \in \mathbf{Z}$.

Proof. Since the function $\mathrm{B}_{(\beta,c_0),\gamma}(\eta)$ of the previous lemma is an increasing linear function in η, every $k \in \mathbf{Z}$ satisfies the bound $k \leq \mathrm{B}_{(\beta,c_0),\gamma}(\eta)$ for η big enough. □

Let $\beta \in \mathrm{N}_1(\mathcal{X})$. We collect its corresponding counts in a series

$$\mathrm{DT}^{\zeta_\gamma,\eta}_\beta(q) = \sum_{c \in \mathrm{N}_0(\mathcal{X})} \mathrm{DT}^{\zeta_\gamma,\eta}_{(\beta,c)} q^c. \tag{5.3.69}$$

Recall that Theorem 5.1.3 proves the existence of a unique rational function $f_\beta(q)$

associated to any curve class $\beta \in \mathrm{N}_1(\mathcal{X})$, such that its expansion with respect to $\mathrm{L}_{\deg}$ is the generating series of stable pair invariants $\mathrm{PT}(\mathcal{X})_\beta$.

Theorem 5.3.34. Let $\beta \in \mathrm{N}_1(\mathcal{X})$ be a curve class and let $\gamma \notin \mathrm{V}_\beta$ not be a wall for β. Then $\mathrm{DT}_\beta^{\zeta_\gamma}(q)$ is the expansion of $f_\beta(q)$ in $\mathbf{Z}[\mathrm{N}_0(\mathcal{X})]_{\mathrm{L}_\gamma}$.

Proof. The set of walls is finite by Lemma 5.3.22, so we may write $\mathrm{V}_\beta = \{\gamma_i\}_{i=1}^n$ where

$$0 < \gamma_1 < \gamma_2 < \ldots < \gamma_n. \tag{5.3.70}$$

By Proposition 5.3.19, we know that $\mathrm{DT}_\beta^{\zeta_\gamma}(q) \in \mathbf{Z}[\mathrm{N}_0(\mathcal{X})]_{\mathrm{L}_\gamma}$. What remains to be shown is that it is an expansion of $f_\beta(q)$.

Lemmas 5.2.15, 5.3.22, and 5.3.26 prove the claim when $0 < \gamma < \gamma_1$. Moreover, by Lemma 5.3.22 and induction on i, it suffices to prove the claim for $\gamma_+ = \gamma_i + \epsilon$ under the assumption that the claim is true for $\gamma_- = \gamma_i - \epsilon$. By Lemma 5.3.27 then, there is up to scale a unique class $c_{\gamma_i} \in \mathrm{N}_0(\mathcal{X})$ such that $\mathrm{L}_{\gamma_i}(c_{\gamma_i}) = 0$, $\mathrm{L}_{\gamma_-}(c_{\gamma_i}) > 0$, and $\mathrm{L}_{\gamma_+}(c_{\gamma_i}) < 0$.

By induction, the series $\mathrm{DT}_\beta^{\zeta_{\gamma_-}}(q)$ is the expansion of $f_\beta(q)$ in $\mathbf{Z}[\mathrm{N}_0(\mathcal{X})]_{\mathrm{L}_{\gamma_-}}$. By Proposition 5.3.31, we have the equality of coefficients

$$\mathrm{DT}^{\zeta_{\gamma_\pm}}_{(\beta, c_0 + k c_{\gamma_i})} = \mathrm{DT}^{\zeta_{\gamma_i, \pm\infty}}_{(\beta, c_0 + k c_{\gamma_i})}. \tag{5.3.71}$$

Thus it follows by Corollary 5.3.33, that the difference

$$\mathrm{DT}^{\zeta_{\gamma_+}}_{(\beta, c_0 + k c_{\gamma_i})} - \mathrm{DT}^{\zeta_{\gamma_-}}_{(\beta, c_0 + k c_{\gamma_i})} \tag{5.3.72}$$

is quasi-polynomial in k. Finally, by Lemma 2.5.16 we may conclude that the series $\mathrm{DT}_\beta^{\zeta_{\gamma_+}}(q)$ is the re-expansion of $f_\beta(q)$ in $\mathbf{Z}[\mathrm{N}_0(\mathcal{X})]_{\mathrm{L}_{\gamma_+}}$. This completes the proof. □

5.3.4 Recovering Bryan–Steinberg invariants

We relate the end product of the wall-crossing, $(\mathtt{T}_{\zeta_{\gamma,\eta}}, \mathtt{F}_{\zeta_{\gamma,\eta}})$-pairs as $\gamma \to \infty$, to stable pairs relative the crepant resolution $f\colon \mathrm{Y} \to \mathrm{X}$. By Theorem 5.3.34 from the previous section, this completes the proof of the crepant resolution conjecture.

We introduce and recall some notation from section 2.4. Let $\Psi = \Phi^{-1}$ denote the inverse to the McKay equivalence $\Phi\colon \mathrm{D}(\mathrm{Y}) \to \mathrm{D}(\mathcal{X})$. We have $\Psi(\mathrm{Coh}(\mathcal{X})) = \mathrm{Per}(\mathrm{Y}/\mathrm{X})$ by Theorem 2.4.20. To ease notation we write $\mathrm{Per}_{\le 1}(\mathrm{Y}) := \Psi(\mathrm{Coh}_{\le 1}(\mathcal{X}))$.

By Proposition 4.1.16, an f-stable pair in the sense of [BS16] is equivalent to a $(\mathtt{T}_f, \mathtt{F}_f)$-pair in the abelian category $\mathtt{A}_\mathrm{Y} = \langle \mathcal{O}_\mathrm{Y}[1], \mathrm{Coh}_{\le 1}(\mathrm{Y})\rangle_{\mathrm{ex}}$. Here $(\mathtt{T}_f, \mathtt{F}_f)$ is a torsion pair on $\mathrm{Coh}_{\le 1}(\mathrm{Y})$, where

$$\mathtt{T}_f = \{\mathrm{F} \in \mathrm{Coh}_{\le 1}(\mathrm{Y}) \mid \mathbf{R}f_*\mathrm{F} \in \mathrm{Coh}_0(\mathrm{X})\} \tag{5.3.73}$$

and $\mathtt{F}_f = \mathtt{T}_f^{\perp}$. For a class $(\beta, n) \in \mathrm{N}_{\leq 1}(\mathrm{Y}) = \mathrm{N}_1(\mathrm{Y}) \oplus \mathbf{Z}$, let $\underline{\mathrm{P}}_{\mathrm{BS}}(\beta, n)$ denote the moduli stack of f-stable pairs of class $[\mathrm{G}] = (\beta, n) \in \mathrm{N}_{\leq 1}(\mathrm{Y})$. It is a $\mathbf{C}^*$-gerbe over its coarse space by Propositions 4.2.12 and 4.2.20. Finally, recall from Proposition 4.2.2 that the McKay equivalence induces an isomorphism $\Phi\colon \mathfrak{Mum}_{\mathrm{Y}} \to \mathfrak{Mum}_{\mathcal{X}}$.

The main result in this section is the following proposition.

Proposition 5.3.35. Let $(\beta, n) \in \mathrm{N}_{\leq 1}(\mathrm{Y})$. Restriction induces an isomorphism

$$\Phi|\colon \underline{\mathrm{P}}_{\mathrm{BS}}(\beta, n) \cong \underline{\mathrm{P}}_{\zeta_\gamma}(\Phi(\beta, n)) \tag{5.3.74}$$

provided that $\gamma > \gamma'$ for any wall γ' in the finite set V_β.

The proof occupies the remainder of this section. We begin by introducing a torsion pair on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ which is the limit of the torsion pairs $(\mathtt{T}_{\zeta_{\gamma,\eta}}, \mathtt{F}_{\zeta_{\gamma,\eta}})$ as $\gamma \to \infty$.

Definition 5.3.36. Let $\mathtt{T}_\infty \subset \mathrm{Coh}_{\leq 1}(\mathcal{X})$ denote the subcategory of sheaves T such that if $\mathrm{T} \twoheadrightarrow \mathrm{Q}$ is a surjection in $\mathrm{Coh}_{\leq 1}(\mathcal{X})$, then either $\mathrm{Q} \in \mathrm{Coh}_0(\mathcal{X})$ or $\mathrm{ch}_1(\Psi \mathrm{Q}) \cdot \omega \cdot \mathrm{A} < 0$.

Let $\mathtt{F}_\infty \subset \mathrm{Coh}_{\leq 1}(\mathcal{X})$ denote the subcategory of sheaves F such that if $\mathrm{S} \hookrightarrow \mathrm{F}$ is an injection in $\mathrm{Coh}_{\leq 1}(\mathcal{X})$, then S is pure of dimension one and $\mathrm{ch}_1(\Psi \mathrm{S}) \cdot \omega \cdot \mathrm{A} \geq 0$.

Lemma 5.3.37. The pair $(\mathtt{T}_\infty, \mathtt{F}_\infty)$ defines a torsion pair on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$.

Proof. Clearly $\mathtt{T}_\infty$ is closed under quotients. Consider an extension diagram in $\mathrm{Coh}_{\leq 1}(\mathcal{X})$

$$\begin{tikzcd} \mathrm{A} \arrow[r, hook, "i"] \arrow[d, two heads] & \mathrm{B} \arrow[r, two heads] \arrow[d, two heads, "f"] & \mathrm{C} \arrow[d, dashed, two heads] \\ \mathrm{im}(s) \arrow[r, hook] & \mathrm{Q} \arrow[r, two heads] & \mathrm{Q}/\mathrm{im}(s) \end{tikzcd} \tag{5.3.75}$$

where $\mathrm{A}, \mathrm{C} \in \mathtt{T}_\infty$, where $f\colon \mathrm{B} \twoheadrightarrow \mathrm{Q}$ is a given surjection, and $s = f \circ i$. We deduce that $\mathrm{B} \in \mathtt{T}_\infty$. Since $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ is noetherian, the claim follows by Lemma 2.1.17. □

Lemma 5.3.38. Let $(\beta, c) \in \mathrm{N}_{\leq 1}(\mathcal{X})$. There exists $\mathrm{M}_\beta \in \mathbf{R}$ such that if $\gamma > \mathrm{M}_\beta$, then an object $\mathrm{E} \in \mathtt{A}$ of class $(-1, \beta, c)$ is a $(\mathtt{T}_\infty, \mathtt{F}_\infty)$-pair if and only if it is a $\zeta_{\gamma,\eta}$-pair.

Proof. If $\mathrm{G} \in \mathrm{Coh}_{\leq 1}(\mathcal{X})$ is a sub- or quotient object of E in $\mathtt{A}$, then $0 \leq \beta_{\mathrm{G}} \leq \beta$. Set

$$\mathrm{M}_\beta = \max\left\{ -\frac{\deg(\beta' \cdot \mathrm{A})}{|\,\mathrm{ch}_2(\Psi(\beta' \cdot \mathrm{A})) \cdot \omega|_{\mathrm{Y}}} \;\Big|\; 0 < \beta' \leq \beta \right\}, \tag{5.3.76}$$

which exists by Lemma 2.1.38. The maximum runs over the finite set of γ-walls for β by Lemma 5.3.22. Thus for $\gamma > \mathrm{M}_\beta$ the notion of $\zeta_{\gamma,\eta}$-pair is unchanged. We see that $\mathrm{G} \in \mathtt{T}_{\zeta_{\gamma,\eta}}$ if and only if $\mathrm{G} \in \mathtt{T}_\infty$ whenever $\gamma > \mathrm{M}_\beta$. Similarly, membership of $\mathtt{F}_{\zeta_{\gamma,\eta}}$ and $\mathtt{F}_\infty$ is equivalent. The claim follows. □

There is a standard exact sequence for objects in $\mathrm{Coh}(\mathcal{X})$ induced by Φ.

Lemma 5.3.39. Let $G \in \mathrm{Coh}(\mathcal{X})$. The sequence

$$0 \to \Phi(H^{-1}(\Psi G)[1]) \to G \to \Phi(H^0(\Psi G)) \to 0 \tag{5.3.77}$$

is exact in $\mathrm{Coh}(\mathcal{X})$.

Proof. The McKay equivalence sends G to $\Psi(G) \in \mathrm{Per}(Y) \subset D^{[-1,0]}(Y)$. Taking cohomology with respect to the standard t-structure yields the exact triangle

$$H^{-1}(\Psi G)[1] \to \Psi(G) \to H^0(\Psi G) \tag{5.3.78}$$

in $D(Y)$. But this is precisely the unique short exact sequence induced by the perverse torsion pair on $\mathrm{Per}(Y)$. Hence (5.3.78) is an exact sequence in $\mathrm{Per}(Y)$. By applying the inverse Φ, it follows that (5.3.77) is an exact sequence in $\mathrm{Coh}(\mathcal{X})$. $\square$

The following key result relates the torsion pairs $(\mathtt{T}_\infty, \mathtt{F}_\infty)$ on $\mathrm{Coh}_{\leq 1}(\mathcal{X})$ and $(\mathtt{T}_f, \mathtt{F}_f)$ on $\mathrm{Coh}_{\leq 1}(Y)$ under the McKay equivalence. Some care is needed, because the filtration by dimension of support $\Phi(N_{\leq 1}(Y)) = N_{\mathrm{mr}}(\mathcal{X}) \subsetneq N_{\leq 1}(\mathcal{X})$ is not preserved by Φ.

Lemma 5.3.40. We have

$$\mathtt{T}_f = \Psi(\mathrm{Coh}_0(\mathcal{X})) \cap \mathrm{Coh}(Y). \tag{5.3.79}$$

Moreover, we have the torsion pair decomposition and identifications

$$\mathtt{T}_\infty = \Big\langle \Phi\big(\mathrm{Per}_{\leq 1}(Y) \cap \mathrm{Coh}(Y)[1]\big), \Phi(\mathtt{T}_f) \Big\rangle, \tag{5.3.80}$$

$$\mathtt{F}_\infty = \Phi\big(\mathrm{Per}_{\leq 1}(Y) \cap \mathrm{Coh}(Y) \cap \mathtt{T}_f^\perp\big). \tag{5.3.81}$$

Recall that $\mathtt{T}_f^\perp = \{F \in \mathrm{Coh}_{\leq 1}(Y) \mid \mathrm{Hom}(T, F) = 0 \text{ for all } T \in \mathtt{T}_f\}$.

Proof. The inclusion $\mathtt{T}_f \supseteq \Psi(\mathrm{Coh}_0(\mathcal{X})) \cap \mathrm{Coh}(Y)$ follows from the definition of $\mathtt{T}_f$. For the reverse inclusion, let $T \in \mathtt{T}_f$. Since $\mathtt{T}_f \subset \mathrm{Per}(Y)$, we have $\Phi(T) \in \mathrm{Coh}(\mathcal{X})$. Every one-dimensional component of the support of T is contracted to a point by f, since otherwise its image under f would be a one-dimensional component in the support of $\mathbf{R}f_*(T)$. It follows that T is supported over finitely many points of X, and so $\Phi(T) \in \mathrm{Coh}_0(\mathcal{X})$. This proves (5.3.79).

Let $T \in \mathrm{Per}(Y) \cap \mathrm{Coh}(Y)[1]$. If $T \twoheadrightarrow T'$ with $T' \in \mathrm{Per}(Y)$, then $T' \in \mathrm{Coh}(Y)[1]$. Hence $\mathrm{ch}_1(T') \cdot \omega \cdot A \leq 0$, with equality if and only if $T' \in \mathrm{Coh}_{\leq 1}(Y)$. In that case, the support of T' must be contracted to dimension zero by f, and so $\Phi(T') \in \mathrm{Coh}_0(\mathcal{X})$. This proves that $T \in \mathtt{T}_\infty$, and so $\Phi(\mathrm{Per}(Y) \cap \mathrm{Coh}(Y)[1]) \subset \mathtt{T}_\infty$.

Thus we know that $\langle\Phi(\mathrm{Per}(Y)\cap\mathrm{Coh}(Y)[1]),\Phi(\mathtt{T}_f)\rangle\subseteq\mathtt{T}_\infty$. For the reverse inclusion, let $G\in\mathtt{T}_\infty$, and let $F_0=H^0(\Psi G)$ and $F_{-1}=H^{-1}(\Psi G)$. By Lemma 5.3.39, we have the short exact sequence

$$0\to\Phi F_{-1}[1]\to G\to\Phi F_0\to 0 \tag{5.3.82}$$

in $\mathrm{Coh}(\mathcal{X})$.

The surjection $G\twoheadrightarrow\Phi F_0$ implies that $\Phi F_0\in\mathtt{T}_\infty$. Thus either $\Phi F_0\in\mathrm{Coh}_0(\mathcal{X})$ or $\mathrm{ch}_1(F_0)\cdot\omega\cdot A<0$. But $\mathrm{ch}_1(F_0)$ is effective, which implies $\mathrm{ch}_1(F_0)\cdot\omega\cdot A\geq 0$, and thus $\Phi F_0\in\mathrm{Coh}_0(\mathcal{X})$. By (5.3.79) then $F_0\in\mathtt{T}_f$, and so the decomposition of G in (5.3.82) is the torsion pair decomposition claimed in (5.3.80).

For (5.3.81), let $G\in\mathtt{F}_\infty$, and write $F=H^{-1}(\Psi G)$. By definition of $\mathtt{F}_\infty$, we have $\Phi F\in\mathrm{Coh}_1(\mathcal{X})[-1]$ and $\mathrm{ch}_1(F)\cdot\omega\cdot A\leq 0$. But $\mathrm{ch}_1(F)$ is effective, hence $F\in\mathrm{Coh}_{\leq 1}(Y)$. Again it follows that F is contracted, hence $\Phi F\in\mathrm{Coh}_0(\mathcal{X})$. But $G\in\mathtt{F}_\infty$ implies $F=0$, and thus $\Psi G\in\mathrm{Coh}(Y)$. Finally, equation (5.3.79) implies that $\mathrm{Hom}(\mathtt{T}_f,F)=\mathrm{Hom}(\Phi(\mathtt{T}_f),\Phi(F))=0$ and so $G\in\Phi(\mathrm{Per}(Y)\cap\mathrm{Coh}(Y)\cap\mathtt{T}_f^\perp)$.

If $G\in\mathrm{Per}_{\leq 1}(Y)\cap\mathrm{Coh}(Y)\cap\mathtt{T}_f^\perp$, then $\mathrm{Hom}(\mathtt{T}_f,G)=\mathrm{Hom}(\mathrm{Coh}(Y)[1],G)=0$, and so by the above $\Phi G\in\mathtt{T}_\infty^\perp\cap\mathrm{Coh}_{\leq 1}(\mathcal{X})=\mathtt{F}_\infty$. This completes the proof. □

Finally, *multi-regular* $(\mathtt{T}_\infty,\mathtt{F}_\infty)$-pairs on $\mathcal{X}$ are identified with f-stable pairs on Y under the McKay equivalence. We prove both implications separately.

Lemma 5.3.41. If $E\in\mathtt{A}=\langle\mathcal{O}_\mathcal{X}[1],\mathrm{Coh}_{\leq 1}(\mathcal{X})\rangle$ is a $(\mathtt{T}_\infty,\mathtt{F}_\infty)$-pair of multi-regular curve class $\beta_E\in N_{\mathrm{mr}}(\mathcal{X})$, then $\Psi(E)$ is an f-stable pair.

Proof. Writing E as an iterated extension of objects $\mathcal{O}_\mathcal{X}[1]$ and $E_1,\dots,E_n$ with $E_i\in\mathrm{Coh}_{\leq 1}(\mathcal{X})$ shows that $H^{-1}(\Psi E)$ has rank one, that $H^0(\Psi E)\in\mathrm{Coh}_{\leq 1}(Y)$, and that all other cohomology sheaves of ΨE vanish.

We claim that $H^{-1}(\Psi E)$ is torsion free. Let $T\hookrightarrow H^{-1}(\Psi E)$ be the torsion part of the sheaf $H^{-1}(\Psi E)$. We have an inclusion $\mathbf{R}^0f_*T\hookrightarrow\mathbf{R}^0f_*(H^{-1}(\Psi E))$ and for any $T'\in\mathrm{Coh}(Y)\cap\mathrm{Per}(Y)$ an inclusion $\mathrm{Hom}(T'[1],T[1])\hookrightarrow\mathrm{Hom}(T'[1],H^{-1}(E)[1])$. The codomain of both of these inclusions vanishes since $H^{-1}(E)[1]\in\mathrm{Per}(Y)$, and so we conclude that $T[1]\in\mathrm{Per}(Y)$.

By Lemma 5.3.40, we find $\Phi(T[1])\in\mathtt{T}_\infty$. Since E is a $(\mathtt{T}_\infty,\mathtt{F}_\infty)$-pair, this implies that $\mathrm{Hom}(\Phi T[1],E)=0$. But we have a chain of inclusions

$$\mathrm{Hom}(T,T)\hookrightarrow\mathrm{Hom}(T,H^{-1}(\Psi E))\hookrightarrow\mathrm{Hom}(T[1],\Psi E)=\mathrm{Hom}(\Phi T[1],E)=0 \tag{5.3.83}$$

forcing $T=0$. We conclude that $H^{-1}(\Psi E)$ is torsion free as claimed.

It follows that $H^{-1}(\Psi E)$is of the form $I_C(D)$ for some one-dimensional scheme $C\subset Y$ and some divisor D. But since β_E is multi-regular we have $c_1(\Psi E)=0$, and so $c_1(H^0(\Psi E))=[D]$ by the triangle in equation (5.3.78).

By Lemma 5.3.39, the sheaf $\mathrm{H}^0(\Psi\mathrm{E})$ is perverse. Since E is a $(\mathtt{T}_\infty, \mathtt{F}_\infty)$-pair, we must have $\Phi(\mathrm{H}^0(\Psi\mathrm{E})) \in \mathtt{T}_\infty$. By Lemma 5.3.40, this implies $\mathrm{H}^0(\Psi\mathrm{E})) \in \mathtt{T}_f$, and so in particular $\mathrm{D} = 0$ and $\mathrm{H}^{>0}(\mathrm{Y}, \mathrm{H}^0(\Psi\mathrm{E})) = 0$. The criterion of Lemma 4.1.16 then implies that $\Psi\mathrm{E}$ has the form $(\mathcal{O}_\mathrm{Y} \to \mathrm{F})$ for some one-dimensional sheaf F on Y. Now for any $\mathrm{T} \in \mathtt{T}_f$, we have $\mathrm{Hom}(\mathrm{T}, \mathrm{F}) = \mathrm{Hom}(\mathrm{T}, \Psi\mathrm{E}) = \mathrm{Hom}(\Phi\mathrm{T}, \mathrm{E}) = 0$, using Lemma 5.3.40, and so $\mathrm{F} \in \mathtt{F}_f$. This proves that $\Psi\mathrm{E}$ is an f-stable pair. □

We now prove the reverse implication.

Lemma 5.3.42. If $\mathrm{I} = (s\colon \mathcal{O}_\mathrm{Y} \to \mathrm{F})$ is an f-stable pair, then $\Phi\mathrm{I}$ is a $(\mathtt{T}_\infty, \mathtt{F}_\infty)$-pair.

Proof. By [BS16, Prop. 18], we have $\mathrm{F} \in \mathrm{Per}_{\le 1}(\mathrm{Y})$. In particular, applying Φ to the triangle $\mathcal{O}_\mathrm{Y} \to \mathrm{F} \to \mathrm{I}$ in $\mathrm{D}(\mathrm{Y})$ shows that $\Phi\mathrm{I}$ is the cone of the map $\mathcal{O}_\mathcal{X} \to \Phi\mathrm{F}$ where $\Phi\mathrm{F} \in \mathrm{Coh}_{\le 1}(\mathcal{X})$. By Lemma 5.3.40, if $\mathrm{T} \in \mathtt{T}_\infty$, then $\mathrm{H}^0(\Psi\mathrm{T}) \in \mathtt{T}_f$. This implies that

$$\mathrm{Hom}(\mathrm{T}, \Phi\mathrm{I}) = \mathrm{Hom}(\mathrm{T}, \Phi\mathrm{F}) = \mathrm{Hom}(\Psi\mathrm{T}, \mathrm{F}) = \mathrm{Hom}(\mathrm{H}^0(\Psi\mathrm{T}), \mathrm{F}) = 0,$$

because $\mathrm{H}^i(\mathrm{T}, \mathcal{O}_\mathcal{X}) = \mathrm{H}^{3-i}(\mathcal{X}, \mathrm{T}) = 0$ for $i = 0, 1$ since $\dim(\mathrm{T}) \le 1$.

If $\mathrm{F} \in \mathtt{F}_\infty$, then by Lemma 5.3.40, we have $\Psi\mathrm{F} \in \mathrm{Coh}(\mathrm{Y}) \cap \mathtt{T}_f^\perp$. This implies that

$$\mathrm{Hom}(\Phi\mathrm{I}, \mathrm{F}) = \mathrm{Hom}(\mathrm{I}, \Psi\mathrm{F}) = \mathrm{Hom}(\mathrm{H}^0(\mathrm{I}), \Psi\mathrm{F}) = 0,$$

because $\mathrm{H}^0(\mathrm{I}) \in \mathrm{T}_f$ since I is an f-stable pair. This completes the proof. □

As a consequence, we may define the generating function of f-stable pair invariants of class $\beta \in \mathrm{N}_{\mathrm{mr}}(\mathcal{X})$ as the generating function of $(\mathtt{T}_\infty, \mathtt{F}_\infty)$-pairs of class β, which is nothing but the generating function of $\zeta_{\gamma,\eta}$-pairs of class β with $\gamma \gg 0$.

Collecting our previous results, we prove the crepant resolution conjecture.

Theorem 5.3.43. Let $\mathcal{X}$ be a three-dimensional Calabi-Yau orbifold satisfying the hard Lefschetz condition with projective coarse moduli space. For each multi-regular curve class $\beta \in \mathrm{N}_{1,\mathrm{mr}}(\mathcal{X})$ there exists a rational function $f_\beta(q)$ such that

1. the expansion of $f_\beta(q)$ with respect to $\mathrm{L}_{\mathrm{deg}}$ is the series $\mathrm{PT}(\mathcal{X})_\beta$,

2. the expansion of $f_\beta(q)$ with respect to L_γ is the series $\mathrm{PT}_f(\mathrm{Y}/\mathrm{X})_\beta$,

where $\gamma > \gamma_r$ and $\mathrm{V}_\beta = \{\gamma_1 < \gamma_2 < \ldots < \gamma_r\}$ is the set of γ-walls for β.

Proof. This follows from Theorem 5.3.34 and Proposition 5.3.35. For an illustration of the γ-wall-crossing, see diagram 5.1.2. □

Remark 5.3.44. The corresponding result with Behrend weighted Euler characteristics replaced by ordinary Euler characteristics, goes through without any change. Moreover, via a compactification argument, this result can be extended to *quasi-projective* CY3 orbifolds $\mathcal{X}'$, provided that $\mathrm{Pic}(\mathcal{X}')$ is finitely generated so as to assure that we can find a (and hence any) compactification satisfying $\mathrm{H}^1(\overline{\mathcal{X}}, \mathcal{O}_{\overline{\mathcal{X}}}) = 0$; this will be treated in [BCR]. An example of such $\mathcal{X}'$ is given by the class of toric CY3 orbifolds.

Note however, that in order to extend the (Behrend weighted) result for DT type invariants to quasi-projective CY3 orbifolds, an extension is needed of the Behrend function identities of [Tod16a, Thm. 2.6] to the non-compact setting. Although these identities are widely believed to hold in the non-compact case too, pursuing their proof lies beyond the scope of this thesis, and beyond the scope of [BCR]. Once established, however, the methods of the above proof should extend the crepant resolution conjecture to the case of toric CY3 orbifolds satisfying the hard Lefschetz condition, thus providing a valid interpretation of the incorrect generating series claim of [Ros17] in terms of a re-expansion of rational functions.

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